

## Restricted version of the infinitesimal Hilbert 16th problem\*

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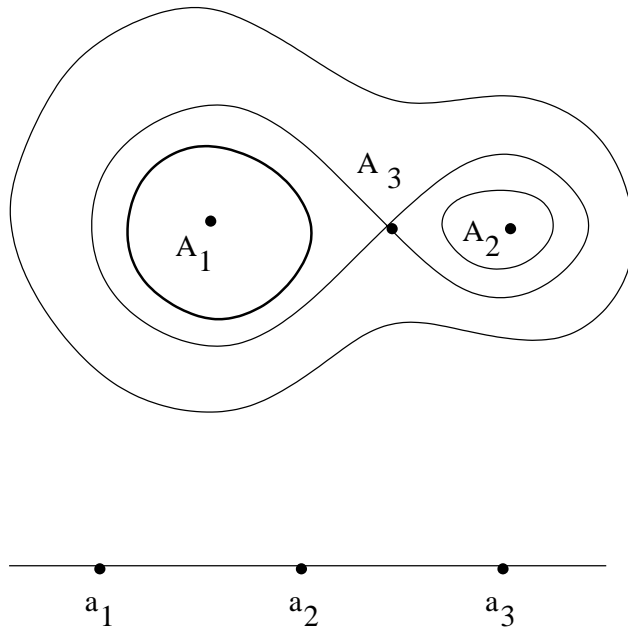
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# 1 Introduction

## 1.1 Restricted Infinitesimal Hilbert 16th Problem

The original Infinitesimal Hilbert 16th Problem is stated as follows. Consider a real polynomial  $H$  in two variables of degree  $n + 1$ . The space of all such polynomials is denoted by  $\mathcal{H}_n$ .

Connected components of closed level curves of  $H$  are called *ovals* of  $H$ . Ovals form continuous families, see Fig. 1. Fix one family of ovals, say  $\Gamma$ , and denote by  $\gamma_t$  an oval of this family that belongs to the level curve  $\{H = t\}$ .



Families of ovals; an oval around  $A_1$  that belongs to the level curve  $H=H(A_2)$  is distinguished.

Figure 1:

Consider a polynomial one-form

$$\omega = Adx + Bdy$$

with polynomial coefficients of degree at most  $n$ . The set of all such forms is denoted by  $\Omega_n$ . The main object to study below is the integral

$$I(t) = \int_{\gamma_t} \omega. \quad (1.1)$$

**Infinitesimal Hilbert 16th Problem .** *Let  $H$  and  $\omega$  be as above. Find an upper bound of the number of isolated real zeros of integral (1.1) for a polynomial  $H \in \mathcal{H}_n$  and any family  $\Gamma$  of real ovals of  $H$ . The estimate should be uniform in  $\omega$  and  $H$ , thus depending on  $n$  only.*

This problem stated more than 30 years ago is not yet solved. The existence of such a bound was proved by A.N.Varchenko [21] and A.G.Khovanskii [10]. A weaker version of the problem is called *restricted*. In order to formulate it we need the following

**1.1 Definition** A polynomial  $H \in \mathcal{H}_n$  is *ultra-Morse* provided that it has  $n^2$  complex Morse critical points with pairwise distinct critical values, and the sum  $h$  of its higher order terms has no multiple linear factors.

Denote by  $\mathcal{U}_n$  the set of all ultra-Morse polynomials in  $\mathcal{H}_n$ . The complement to this set is denoted by  $\Sigma_n$  and called *the discriminant set*. The integral (1.1) may be identically zero. The following theorem shows that for ultra-Morse polynomials this may happen by a trivial reason only.

**1.2 Theorem (Exactness theorem [5, 6, 18]).**

*Let  $H$  be a real ultra-Morse polynomial of degree higher than 2. Let the integral (1.1) be identically zero for some family of real ovals of the polynomial  $H$ . Then the form  $\omega$  is exact:  $\omega = df$ .*

Denote by  $\Omega_n^*$  the set of all non-exact polynomial one-forms from  $\Omega_n$ .

**Restricted version of the Infinitesimal Hilbert 16th Problem .** *For any compact set  $\mathcal{K} \subset U_n$  find an upper bound of the number of all real zeros of the integral (1.1) over the ovals of the polynomial  $H \in \mathcal{K}$ . The bound should be uniform with respect to  $H \in \mathcal{K}$  and  $\omega \in \Omega_n^*$ . It may depend on  $n$  and  $\mathcal{K}$  only.*

This problem is solved in the present paper, and the explicit upper bound is given in the next subsection.

## 1.2 Main results

To measure a gap between a compact set  $\mathcal{K} \subset U_n$  and the discriminant set  $\Sigma_n$ , let us first normalize ultra-Morse polynomials by an affine transformation in the target space. This transformation does not change the ovals of  $H$ , thus the number of zeros of the integral (1.1) remains unchanged.

Say that two polynomials  $G$  and  $H$  are equivalent iff

$$G = aH + b, \quad a > 0, \quad b \in \mathbb{C}.$$

**1.3 Definition** A polynomial is *balanced* if all its complex critical values belong to a disk of radius 2 centered at zero, and there is no smaller disk that contains all the critical values.

**1.4 Remark** Any polynomial with at least two distinct critical values is equivalent to one and unique balanced polynomial. If the initial polynomial has real coefficients, then so does the corresponding balanced polynomial.

Define two positive functions on  $\mathcal{U}_n$  such that at least one of them tends to zero as  $H$  tends to  $\Sigma_n$ . For any compact set  $\mathcal{K} \subset \mathcal{U}_n$  the minimal values of these functions on  $\mathcal{K}$  form a vector in  $\mathbb{R}^+ \times \mathbb{R}^+$  that is taken as a size of the gap between  $\mathcal{K}$  and  $\Sigma_n$ .

**1.5 Definition** For any  $H \in \mathcal{U}_n$  let  $c_1(H)$  be  $n$  multiplied by the smallest distance between two lines in the locus of  $h$ , the higher order form of  $H$ . The distance between two lines is taken in sense of Fubini-Study metric on the projective line  $\mathbb{C}P^1$ . Let  $c'(H) = \min(c_1(H), 1)$ .

Denote by  $\mathcal{V}_n$  the set of all polynomials with more than one critical value and more than one line in the locus of the higher order homogeneous form. By Definition 1.1,  $\mathcal{U}_n \subset \mathcal{V}_n$ .

**1.6 Definition** For any  $H \in \mathcal{V}_n$ , let  $G$  be the balanced polynomial equivalent to  $H$ . Let  $c_2(H)$  be the minimal distance between two critical values of  $G$  multiplied by  $n^2$ . Let  $c''(H) = \min(c_2(H), 1)$ .

Note that inequality  $c'(H)c''(H) > 0$  is equivalent to the statement that  $H$  is ultra-Morse.

In what follows, we deal with balanced ultra-Morse polynomials only. This may be done without loss of generality: any ultra-Morse polynomial is equivalent to a balanced one; equivalent polynomials have the same number of zeros of the integral (1.1) over the same family of ovals.

**Theorem A.** *Let  $H$  be a real ultra-Morse polynomial of degree  $n + 1$ . Let  $\Gamma = \{\gamma_t\}$  be an arbitrary continuous family of real ovals of  $H$ . There exists a universal positive  $c$  such that the integral (1.1) has at most  $(1 - \log c'(H))e^{\frac{c}{c''(H)}n^4}$  isolated zeros.*

**Appendix.** The statement of Theorem A holds with  $c = 5.000$ .

An approach to the Infinitesimal Hilbert 16th Problem itself presented below motivates the following complex counterpart of Theorem A, namely, Theorem B that gives an estimate of the number of zeros of the integral (1.1) in the complex domain. Consider an ultra-Morse polynomial  $H$  and let

$$\nu = \nu(H) := \frac{c''(H)}{4n^2} \tag{1.2}$$

Fix any real noncritical value  $t_0$  of  $H$ ,

$$|t_0| < 3,$$

whose distance to the complex critical values of  $H$  is no less than  $\nu$ . Consider a real oval  $\gamma_0 \subset \{H = t_0\}$ . We suppose that such an oval exists. Let  $a = a(t_0) < t_0 < b(t_0) = b$  (or  $a(H, t_0)$ ,  $b(H, t_0)$  for variable  $H$ ) be the nearest real critical values of  $H$  to the left and to the right from  $t_0$  respectively; or  $-\infty, +\infty$  if there are none. Denote by  $\sigma(t_0)$  the interval  $(a(t_0), b(t_0))$  and let  $\Gamma(\gamma_0)$  be the continuous family of ovals that contains  $\gamma_0$  :

$$\Gamma(\gamma_0) = \{\gamma(t) \mid t \in \sigma(t_0), \gamma(t_0) = \gamma_0\}. \tag{1.3}$$

The following cases for  $(a, b) = \sigma(t_0)$  are possible:

$$(a, b), b > a; (a, +\infty); (-\infty, b).$$

If  $\lim_{t \rightarrow a} \text{top } \gamma(t)$  contains a critical point of  $H$ , then  $a$  is a logarithmic branch point of  $I$ . If not,  $a$  is called an apparent singularity. The same for  $b$ . Let

$$W = W(t_0, H)$$

be the universal cover over the set of noncritical values of  $H$  with the base point  $t_0$  and the projection  $\pi : W \rightarrow \mathbb{C}$ . Let  $D(t, r)$  be the disk centered at  $t$  of radius  $r$ . Denote by  $a + re^{i\varphi} \in W$  a point represented by a curve  $\Gamma_1 \Gamma_2 \subset W$ , where  $\Gamma_1$  is an oriented segment from  $t_0$  to  $t_1 = a + r \in \sigma(t_0)$ ,  $\Gamma_2 = \{a + re^{i\theta} \mid \theta \in [0, \varphi]\}$ ;  $\Gamma_2$  is oriented from  $t_1$  to  $t$ . In the same way  $b - re^{i\varphi} \in W$  is defined. Let

$$\Pi(a) = \{a + re^{i\varphi} \in W \mid 0 < r \leq \nu, |\varphi| \leq 2\pi\}, \text{ for } a \neq -\infty \quad (1.4)$$

$$\Pi(b) = \{b - re^{i\varphi} \in W \mid 0 < r \leq \nu, |\varphi| \leq 2\pi\}, \text{ for } b \neq +\infty$$

Let

$$D(l, a) = \{a + re^{i\varphi} \in W \mid a + re^{\frac{i\varphi}{l}} \in \Pi(a)\}$$

$$D(l, b) = \{b - re^{i\varphi} \in W \mid b - re^{\frac{i\varphi}{l}} \in \Pi(b)\}$$

Let  $DP_R = DP_R(H, t_0)$  be the disk of radius  $R$  in the Poincaré metric of  $W$  centered at  $t_0$ . Denote

$$S_t = \{H = t\} \subset \mathbb{C}^2.$$

For any real polynomial  $H$ , the choice of a cycle  $\gamma_0$  determines a family of ovals (1.3) over which the integral (1.1) is taken. When we want to specify this choice we write  $I_{H, \gamma_0}$  or  $I_H$  instead of  $I$ . The integral  $I_{H, \gamma_0}$  may be analytically extended not only as a function of  $t$ , but also as a function of  $H$ . We stress that the integral  $I_{H, \gamma_0}(t)$  is taken over the oval  $\gamma_{H, t}(t) \subset S_t$ ; the family of ovals depends continuously on  $H$  and  $t$  and may be extended to complex values of  $t$  as elements of the homology group  $H_1(S_t, \mathbb{Z})$  depending continuously on  $H$  and  $t$ .

An analytic extension of the integral  $I$  to  $W$  is denoted by the same symbol  $I$ . For any positive  $R$  and natural  $l$  denote by  $G = G(l, R, H, t_0)$  the domain

$$G = DP_R(H, t_0) \cup D(l, a(H, t_0)) \cup D(l, b(H, t_0)).$$

**Theorem B.** *For any real ultra-Morse polynomial  $H$ , any real oval  $\gamma_0$  of  $H$ , any natural  $l$  and any positive  $R > \frac{288n^4}{c''(H)}$ , the number of zeros of the integral  $I_{H, \gamma_0}$  in  $G = G(l, R, H, t_0)$ , where  $t_0 = H \mid \gamma_0$ , is estimated as follows:*

$$\#\{t \in G(l, R, H, t_0) \mid I_{H, \gamma_0}(t) = 0\} \leq (1 - \log c'(H)) \cdot (e^{7R} + A^{4800} e^{\frac{481l}{c''(H)}}), \quad A = e^{\frac{n^4}{c''(H)}}. \quad (1.5)$$

The lower bound on  $R$  in the statement of the theorem is motivated by the remark in Subsection 1.7 below.

### 1.3 An approach to a solution of the Infinitesimal Hilbert 16th Problem

**Conjecture.** For any  $n$  there exist  $\delta(n), l(n), R(n)$  with the following property. Let  $H_0$  be an arbitrary real polynomial from  $\mathcal{H}_n$ ,  $t_0$  be its real noncritical value and  $\gamma_0$  be a real oval of  $H_0$  that belongs to  $\{H_0 = t_0\}$  (we suppose that such an oval exists). Let  $I_H$  be the integral (1.1). The integral  $I_H$  depends on  $H$  as a parameter. Let  $t_1 \in \sigma(t_0)$ ,  $I_{H_0}(t_1) = 0$  and  $t(H)$  be a germ of an analytic function defined by the equation  $I_H(t(H)) \equiv 0$ ,  $t(H_0) = t_1$ . The required property is the following. There exists a path  $\lambda \subset \mathcal{H}_n$  depending on  $H_0$  only starting at  $H_0$  and ending at some  $H_1 \in \mathcal{H}_n$  such that:

$$c'(H_1) \geq \delta(n), \quad c''(H_1) \geq \delta(n);$$

$t_0$  is a noncritical value of all the polynomials along the path  $\lambda$ ;

the analytic extension  $t(H_1)$  of the function  $t(H)$  along  $\lambda$  starting at the value  $t_1$  belongs to the domain  $G(l(n), R(n), H_1, t_0)$ .

The conjecture above implies the solution of the Infinitesimal 16th Problem. Indeed, suppose that the conjecture is true. Let

$$N(n) = (1 - \log \delta(n))e^{7R(n) + \frac{c_1 n^4}{\delta(n)} + \frac{c_2 l(n)}{\delta(n)}}, \quad c_1 = 4800, c_2 = 481$$

Then the number of real zeros of integral  $I_{H_0}$  can not exceed  $N(n)$ . If not, any of real zeros of  $I_{H_0}$  would be extended along  $\lambda$  up to a zero of a polynomial  $H_1$  located in  $G = G(l(n), R(n), H_1, t_0)$ . Thus the number of zeros of the integral  $I_{H_1}$  in  $G$  will exceed  $N(n)$ . But Theorem B implies:

$$\#\{t \in G(l(n), R(n), H_1, t_0) \mid I_{H_1, t_0}(t) = 0\} \leq N(n),$$

a contradiction.

### 1.4 Historical remarks

A survey of the history of Infinitesimal Hilbert 16th Problem may be found in [7], and we will not repeat it here. In particular, a much weaker version of Theorem A is claimed there as Theorem 7.7. The first solution to restricted Hilbert problem was suggested in [16]. An explicit upper bound for the same numbers of zeros as in Theorem A was suggested there as a tower of four exponents with coefficients “that may be explicitly written following the proposed constructive solution.” It is unclear how much efforts is needed to write these constants down. Moreover, exponential of a polynomial presented in Theorem A is much simpler (though still very excessive) than the tower of four exponentials.

The result of [16] is a crown of a series of papers [13] - [15]. Solution to the restricted version of the Infinitesimal Hilbert 16th Problem presented there is only one application of a vast theory.

This theory presents an upper bound of the number of zeros of solutions to linear systems of differential equations. Similar results for components of vector solutions to linear systems are obtained. Abelian integrals are considered as solutions to Picard-Fuchs equations.

On the contrary, our presentation is focused on the study of Abelian integrals given by formula (1.1) “as they are” and not as solutions of differential equations.

## 1.5 Quantitative algebraic geometry

Our main tool is Growth-and-Zeros theorem for holomorphic functions stated in the next subsection. It requires, in particular, an upper bound of the integral under consideration. We fix an integrand, say  $w = x^k y^{n-k} dx$ . Depending on a scale in  $\mathbb{C}^2$ , a cycle  $\gamma$  in the integral  $\int_{\gamma} \omega$  may be located in a small or in a large ball. According to this, the integrand will be small or large. We want to estimate the integral at a certain point of the universal cover  $W$  represented by an arc that connects a base point  $t_0$  with some point, say  $t$ , with  $|t| \leq 3$ . To make this restriction meaningful, the scale in the range of the polynomial should be chosen; in other words, the polynomial should be balanced. The argument above shows that it should be also *rescaled* in sense of the following definitions.

**1.7 Definition** The *norm* of a homogeneous polynomial is the maximal value of its module on the unit sphere; this norm is denoted by  $\|h\|_{\max}$ .

**1.8 Definition** A balanced polynomial  $H \in \mathbb{C}[x, y]$  is *rescaled* provided that the norm of its higher order form  $h$  equals one:  $\|h\|_{\max} = 1$ , and the origin is a critical point for  $H$ . Briefly, a balanced rescaled polynomial will be called *normalized*.

**1.9 Remark** Any ultra-Morse polynomial may be transformed to a normalized one by affine transformations in the source and target spaces (not in the unique way). The functions  $c'$  and  $c''$  remain unchanged under such transformations.

**1.10 Definition** We say that the topology of a level curve  $S_t = H^{-1}(t)$  of a polynomial  $H \in \mathcal{H}_n$  is *located in a bidisk*

$$D_{X,Y} = \{(x, y) \in \mathbb{C}^2 \mid |x| \leq X, |y| \leq Y\}$$

provided that the difference  $S_t \setminus D_{X,Y}$  consists of  $n+1 = \deg H$  punctured topological disks, and the restriction of the projection  $(x, y) \mapsto x$  to any of these disks is a biholomorphic map onto  $\{x \in \mathbb{C} \mid X < |x| < \infty\}$ .

**Theorem C [3].** *For a normalized polynomial, the Hermitian basis in  $\mathbb{C}^2$  may be so chosen that the topology of all level curves  $S_t$  for  $|t| \leq 5$  will be located in a bidisk  $D_{X,Y}$  with*

$$X \leq Y \leq (c'(H))^{-14n^3} n^{65n^3} = R_0.$$

This theorem is of independent interest, providing one of the first results in *quantitative algebraic geometry*. On the other hand, it implies upper estimates of Abelian integrals used in the proof of Theorem A and required by the Growth-and-Zeros theorem below.

In what follows, we describe the main ideas of the proof of a simplified version of Theorem A, namely Theorem A1 stated below. It provides an upper bound for the number of zeros of the integral (1.1) on a real segment that is  $\nu$ -distant from critical values of  $H$  and belongs to the disk  $\overline{D}_3 = \{t \mid |t| \leq 3\}$ , thus being distant from infinity; recall that  $\nu = \nu(H)$  is given by (1.2).

By the use of Theorem A1, we get in Section 4 an estimate of the number of zeros of the integral  $I_{H,\gamma_0}$  near the endpoints of  $\sigma(t_0)$ , as well as near infinity (Theorem A2 stated in 1.8).

Together with Theorem A1, this completes the proof of Theorem A. Theorem B is split into two parts. The first one (Theorem B1 stated in 3.1) is proved by extending an upper bound given with the help of Theorem C from a disk  $|t| \leq 5$  into a larger domain. The second one, Theorem B2 stated in 4.6, is proved (in the same place) by the same tools as Theorem A2 that include Petrov method and a so called KRY theorem. The latter one is a recent result in one-dimensional complex analysis [11, 19]. Its improved version is proved by the second author (Yu.S.Ilyashenko) in a separate paper [8] and stated in Section 4. In this form it provides a mighty tool to estimate the number of zeros of analytic functions near logarithmic singularities.

## 1.6 Growth-and-Zeros Theorem for Riemann surfaces

The idea of the proof of Theorem A1 is to consider an analytic extension of the integral (1.1) to the complex domain and to make use of the following Growth-and-Zeros theorem. The definition of the intrinsic diameter used in the statement of the theorem is recalled below. We need the following

**1.11 Definition** Let  $W$  be a Riemann surface,  $\pi : W \rightarrow \mathbb{C}$  be a holomorphic function (called projection) with non-zero derivative. Let  $\rho$  be the metric on  $W$  lifted from  $\mathbb{C}$  by projection  $\pi$ . Let  $U \subset W$  be a connected domain, and  $K \subset U$  be a compact set. For any  $p \in U$  let  $\varepsilon(p, \partial U)$  be the supremum of radii of disks centered at  $p$ , located in  $U$  and such that  $\pi$  is bijective on these disks. The  $\pi$ -gap between  $K$  and  $\partial U$ , is defined as

$$\pi\text{-gap}(K, \partial U) = \min_{p \in K} \varepsilon(p, \partial U).$$

**Growth-and-zeros theorem.** *Let  $W, \pi, \rho$  be the same as in Definition 1.11. Let  $U \subset W$  be a domain conformally equivalent to a disk. Let  $K \subset U$  be a path connected compact subset of  $U$  (different from a single point). Suppose that the following two assumptions hold:*

*Diameter condition:*

$$\text{diam}_{\text{int}} K \leq D;$$

*Gap condition:*

$$\pi\text{-gap}(K, \partial U) \leq \varepsilon.$$

*Let  $I$  be a bounded holomorphic function on  $\bar{U}$ . Then*

$$\#\{z \in K | I(z) = 0\} \leq e^{\frac{2D}{\varepsilon}} \log \frac{\max_{\bar{U}} |I|}{\max_K |I|} \quad (1.6)$$

The definition of the intrinsic diameter is well known; yet we recall it for the sake of completeness.

**1.12 Definition** The *intrinsic distance* between two points of a path connected set in a metric space is the infimum of the length of paths in  $K$  that connect these points (if exists). The *intrinsic diameter* of  $K$  is the supremum of intrinsic distances between two points taken over all the pairs of points in  $K$ .



**1.13 Definition** The second factor in the right hand side of (1.6) is called the Bernstein index of  $I$  with respect to  $U$  and  $K$  and denoted  $B_{K,U}(I)$  :

$$B_{K,U}(I) = \log \frac{M}{m}, \quad M = \sup_U |I|, \quad m = \max_K |I|. \quad (1.7)$$

**Proof of the Growth-and-Zeros theorem.** The above theorem is proved in [9] for the case when  $W = \mathbb{C}, \pi = Id$ . In fact, in [9] another version of (1.6) is proved with (1.6) replaced by

$$\#\{z \in K | I(z) = 0\} \leq B_{K,U}(I)e^\rho, \quad (1.8)$$

where  $\rho$  is the diameter of  $K$  in the Poincaré metric of  $U$ . In this case it does not matter whether  $U$  belongs to  $\mathbb{C}$  or to a Riemann surface.

**1.14 Proposition** *Let  $K, U$  be two sets in the Riemann surface  $W$  from Definition 1.11, and let the Diameter and Gap conditions from the Growth-and-Zeros theorem hold. Then the diameter of  $K$  in the Poincaré metric of  $U$  admits the following upper estimate:*

$$\rho \leq 2D/\varepsilon. \quad (1.9)$$

**Proof** By the monotonicity property of the Poincaré metric, the length of any vector  $v$  attached at any point  $p \in K$  is no greater than two times the Euclidean length of  $v$  divided by the  $\pi$ -gap between  $K$  and  $\partial U$ . This implies (1.9)  $\square$

Together with (1.8), this proves (1.6).  $\square$

## 1.7 Theorem A1 and Main lemma

In what follows,  $H$  will be an ultra-Morse polynomial unless the converse stated. Consider a normalized polynomial  $H$ . Let  $a_j$  be its complex critical values,  $j = 1, \dots, n^2$ ;  $\nu, t_0, W$  and  $\pi$  be the same as in 1.2. Let  $I$  be the integral (1.1) as in Theorem A (well defined for  $t = t_0$ ). It admits an analytic extension to  $W$ , which will be denoted by the same symbol  $I$ .

Let  $a = a(t_0), b = b(t_0)$  be the same as in 1.2, and  $\nu$  be from (1.2). Let

$$l(t_0) = \begin{cases} a + \nu & \text{for } a \neq -\infty \\ -3 & \text{for } a = -\infty, \end{cases}$$

$$r(t_0) = \begin{cases} b - \nu & \text{for } b \neq +\infty \\ 3 & \text{for } b = +\infty. \end{cases}$$

Let

$$\sigma(t_0, \nu) = [l(t_0), r(t_0)].$$

We identify  $\sigma(t_0, \nu) \subset \mathbb{C}$  with its lift to  $W$  that contains  $t_0$ .

**Theorem A1.** *In the assumptions at the beginning of the subsection, for any complex form  $\omega \in \Omega_n^*$ ,*

$$\#\{t \in \sigma(t_0, \nu) \mid I(t) = 0\} < (1 - \log c')A^{578}, \quad A = e^{\frac{n^4}{27}}(H). \quad (1.10)$$

This theorem is an immediate corollary of the Growth-and-Zeros theorem and the Main Lemma stated below. Let

$$L^\pm(t_0) = \begin{cases} \{a + \nu e^{\pm i\varphi} \in W \mid \varphi \in [0, 2\pi]\} & \text{for } a \neq -\infty \\ \{-3e^{\pm i\varphi} \in W \mid \varphi \in [0, 2(n+1)\pi]\}, & \text{for } a = -\infty, \end{cases}$$

$$R^\pm(t_0) = \begin{cases} \{b - \nu e^{\pm i\varphi} \in W \mid \varphi \in [0, 2\pi]\} & \text{for } b \neq +\infty \\ \{+3e^{\pm i\varphi} \in W \mid \varphi \in [0, 2(n+1)\pi]\}, & \text{for } b = +\infty, \end{cases}$$

$$\Gamma_a = L^+(t_0) \cup L^-(t_0), \quad \Gamma_b = R^+(t_0) \cup R^-(t_0), \quad \Sigma = \Gamma_a \cup \Gamma_b \cup \sigma(t_0, \nu).$$

**Main Lemma.** *Let  $H$  be a normalized polynomial of degree  $n+1 \geq 3$  with critical values  $a_j : j = 1, \dots, n^2$ ,  $\omega$  be a complex polynomial 1-form of degree no greater than  $n$ . Let  $W, \nu, \Sigma$  be the same as at the beginning of this subsection. Then there exists a path connected compact set  $K \subset W$ ,  $K \supset \Sigma$ ,  $\pi K \subset \overline{D_3}$ , with the following properties:*

$$\text{diam}_{\text{int}} K < 36n^2; \tag{1.11}$$

$$\rho(\pi K, a_j) \geq \nu \text{ for any } j = 1, \dots, n^2. \tag{1.12}$$

Moreover, let  $U$  be the minimal simply connected domain in  $W$  that contains the  $\nu/2$  neighborhood of  $K$ . Then

$$B_{K,U}(I) < (1 - \log c')A^2. \tag{1.13}$$

The Lemma is proved in Section 2. It is used also in the estimate of the number of zeros of the integral in the intervals  $(a, l(t_0))$ ,  $(r(t_0), b)$ . In fact, a much better estimate for the Bernstein index holds:

$$B_{K,U}(I) < \frac{2700n^{18}}{c''(H)} - 30n^6 \log c'(H) := B(n, c', c''). \tag{1.14}$$

Inequality (1.14) is proved in 2.7. Together with the elementary inequality

$$B(n, c', c'') < (1 - \log c')A^2, \tag{1.15}$$

it implies (1.13).

**Proof of Theorem A1.** Let us apply Growth-and-Zeros theorem to the function  $I$  in the domain  $U$  in order to estimate the number of zeros of  $I$  in  $K$ ; note that  $K \supset \sigma(t_0)$ . The intrinsic diameter of  $K$  is estimated from above by (1.11). The gap condition for  $U$  and  $K$  has the form

$$\varepsilon(K, \partial U) = \frac{\nu}{2} = \frac{c''}{8n^2}$$

by the definition of  $U$ . Hence,

$$e^{\frac{2D}{\varepsilon}} < e^{\frac{72n^2}{c''} 8n^2} = A^{576}.$$

The Bernstein index  $B_{K,U}(I)$  is estimated from above in (1.13). By Growth-and-Zeros theorem

$$\#\{t \in \sigma(t_0) \mid I(t) = 0\} < B_{K,U}(I)A^{576} < (1 - \log c')A^{578}.$$

This proves (1.10).  $\square$

The following remark motivates the restriction on  $R$  in Theorem B.

**1.15 Remark** Let  $K$  be the set from the Main Lemma,  $\rho_W K$  be its diameter in the Poincaré metric of  $W$ . Then

$$\rho_W K < (c'')^{-1}288n^4. \quad (1.16)$$

Indeed,  $\rho_W K$  is no greater than the ratio of the double intrinsic diameter of  $K$  divided by its minimal distance to the critical values of  $H$ . Together with (1.11) and (1.12) this implies (1.16). On the other hand, in the proof of Theorem B, we apply Growth-and-Zeros theorem in the case, when the Poincaré disc  $DP_R(H, t_0)$  is large enough, namely, contains the set  $K$ .

## 1.8 Theorem A2 and proof of Theorem A

**Theorem A2.** *Let  $H, t_0, a = a(t_0), b = b(t_0)$  be the same as in the previous subsection. Let  $\omega$  be a real 1-form in  $\Omega_n^*$ . Then, in assumptions of Theorem A1,*

$$\#\{t \in (a, l(t_0)) \cup (r(t_0), b) \mid I(t) = 0\} < (1 - \log c')A^{4800} \quad (1.17)$$

**Proof of Theorem A.** By Theorems A1 and A2

$$\#\{t \in (a, b), I(t) = 0\} < (1 - \log c')A^{578} + (1 - \log c')A^{4800} < 2(1 - \log c')A^{4800}. \quad (1.18)$$

This implies the estimate of the number of zeros given by Theorem A on the interval  $(a, b)$ .

Let  $\sigma' \subset \mathbb{R}$  be the maximal interval of continuity of the family  $\Gamma$  of real ovals that contains  $\gamma_0$ . Then  $\sigma'$  is bounded by a pair of critical values, at most one of them may be infinite. In general, the interval  $\sigma'$  may contain critical values (see Fig.1, which presents a possible arrangement of level curves of  $H$  in this case:  $A_1, A_2, A_3$  are critical points of  $H$ ,  $a_j = H(A_j)$ ,  $a_2 \in \sigma' = (a_1, a_3)$ ). In this case  $\sigma' \neq (a, b) = (a_1, a_2)$ . Let us estimate the number of zeros on  $\sigma'$ . The interval  $\sigma'$  is split into at most  $n^2$  subintervals bounded by critical values. On each subinterval the number of zeros of  $I$  is estimated by (1.18), as before. Therefore, the number of zeros of  $I$  on  $\sigma'$  is less than  $2n^2(1 - \log c')A^{4800} < (1 - \log c')A^{4801}$ . This proves Theorem A.  $\square$

The paper is structured as follows. In Section 2 we prove the Main Lemma modulo two statements: formula for the determinant of periods, and upper estimates of Abelian integrals provided by quantitative algebraic geometry. These two statements are treated in two separate papers by the first author (A.Glutsyuk, [2] and [3] respectively). After this the Main Lemma, as well as Theorem A1, is proved. Theorem A2 is proved in Section 4. Theorem B is proved in Sections 3 and 4. In both sections, the Main Lemma is intensively used. The complete proof of Theorem A ends up in Section 4.

## 2 An upper bound for the number of zeros on a real segment distant from critical values

In this section we prove the Main Lemma and hence Theorem A1. We also prove the Modified Main Lemma, see 2.9 below, and prepare important tools for the proof of other results: Theorems A2, B1 and B2.

### 2.1 Normalized ultra-Morse polynomials; notations

Denote by  $D_r$  a disk  $|t| \leq r$ .

All along this section  $H$  is a real normalized ultra-Morse polynomial of degree  $n + 1 \geq 3$ ,  $\mu = n^2$ ;  $a_1, \dots, a_\mu$  are critical values of  $H$ ,  $\nu$  is the same as in (1.2),  $\varepsilon = \nu/2$ . For  $t$  close to  $a_j$ ,  $\delta_j(t)$  is a local vanishing cycle corresponding to  $a$  on a level curve

$$S_t = \{H = t\};$$

the definition of this cycle is recalled in the next subsection. Denote by  $B = B_H$  the set of all noncritical values of  $H$  :

$$B = \mathbb{C} \setminus \{a_1, \dots, a_\mu\}.$$

Let

$$t_0 \in B \cap (-3, 3),$$

and  $W$  be the universal cover over  $B$  with the base point  $t_0$  and projection

$$\pi : W \rightarrow B.$$

### 2.2 Marked system of vanishing cycles

To begin, we recall well known results and definitions.

**2.1 Lemma (Morse lemma).** *A holomorphic function having a Morse critical point may be transformed to a sum of a nondegenerate quadratic form and a constant term by an analytic change of coordinates near this point.*

**2.2 Corollary** *Consider a holomorphic function in  $\mathbb{C}^2$  having a Morse critical point with a critical value  $a$ . An intersection of a level curve of this function corresponding to a value close to  $a$  with an appropriate neighborhood of the critical point is diffeomorphic to an annulus.*

This annulus may be called a local level curve corresponding to the a critical value  $a$ .

**2.3 Definition** A generator of the first homology group of the local level curve corresponding to  $a$  is called a *local vanishing cycle* corresponding to  $a$ .

A local vanishing cycle is well defined up to change of orientation.

A path  $\alpha_j : [0, 1] \rightarrow \mathbb{C}$  is called *regular* provided that

$$\alpha_j(0) = t_0, \alpha_j(1) = a_j, \alpha_j[0, 1) \subset B \tag{2.1}$$

**2.4 Definition** Let  $\alpha_j$  be a regular path,  $s \in [0, 1]$  be close to 1,  $\delta_j(t)$ ,  $t = \alpha_j(s)$ , be a local vanishing cycle on  $S_t$  corresponding to  $\alpha_j$ . Consider the extension of  $\delta_j$  along the path  $\alpha$  up to a continuous family of cycles  $\delta_j(s)$  in complex level curves  $H = \alpha_j(s)$ . The homology class  $\delta_j = \delta_j(0)$  is called a *cycle vanishing along  $\alpha_j$* .

**2.5 Definition** Consider a set of regular paths  $\alpha_1, \dots, \alpha_\mu$ , see (2.1). Suppose that these paths are not pairwise and self intersected. Then the set of cycles  $\delta_j \in H_1(S_{t_0}, \mathbb{Z})$  vanishing along  $\alpha_j$ ,  $j = 1, \dots, \mu$ , is called a *marked set of vanishing cycles* on the level curve  $H = t_0$ .

**2.6 Definition** Any point  $\hat{t} \in W$  is represented by a class  $[\lambda]$  of curves in  $B$  starting at  $t_0$  and terminating at  $t = \pi\hat{t}$ ; all the curves of the class are homotopic on  $B$ . Any cycle  $\gamma$  from  $H_1(S_{t_0}, \mathbb{Z})$  may be continuously extended over  $\lambda$  as an element of the homology groups of level curves of  $H$ ; the resulting cycle  $\gamma(\hat{t})$  from  $H_1(S_{\hat{t}}, \mathbb{Z})$  is called an *extension* of  $\gamma$  corresponding to  $\hat{t}$ .

Let  $\delta_1, \dots, \delta_\mu$  be a marked set of vanishing cycles. For any cycle  $\delta_l$  from this set, denote by  $W_l$  the Riemann surface of the integral

$$I_l(\hat{t}) = \int_{\delta_l(\hat{t})} \omega,$$

with the base point  $t_0$ . Let  $\pi_l$  be the natural projection  $W \rightarrow W_l$ . Denote by  $D_r(a)$  the disk  $|t - a| \leq r$ .

**2.7 Remark** The Riemann surface  $W_l$  contains the disc  $D_\nu(a)$ .

**2.8 Lemma (Modified Main Lemma).** *The Main Lemma from subsection 1.7 holds true provided that the real oval  $\gamma$  is replaced by a local vanishing cycle  $\delta_l(t)$  close to the corresponding critical value  $a_l$ , and  $\Sigma$  is replaced by the disk  $\overline{D}_\nu(a_l)$ .*

This lemma is proved in 2.8.

### 2.3 Matrix of periods

Consider and fix an arbitrary marked set of vanishing cycles  $\delta_j$ ,  $j = 1, \dots, \mu$ . For any  $\hat{t} \in W$ , let  $\delta_j(\hat{t})$  be the extension of  $\delta_j$  corresponding to  $\hat{t}$ .

**2.9 Definition** Consider a set  $\Omega$  of  $\mu$  forms  $\omega_j$  of the type

$$\omega_i = yx^k y^l dx, \quad k, l \geq 0, \quad k + l \leq 2n - 2 \quad (2.2)$$

$(k, l)$  depends on  $i$ , such that all the forms with  $k + l \leq n - 1$  are included in the set. In what follows, such a set is called *standard*.

A *matrix of periods*  $\mathbb{I} = (I_{ij})$ ,  $1 \leq i \leq \mu$ ,  $1 \leq j \leq \mu$ , is the matrix function defined on  $W$  by the formula:

$$I_{ij}(\hat{t}) = \int_{\delta_j(\hat{t})} \omega_i, \quad \mathbb{I}(\hat{t}) = (I_{ij}(\hat{t})) \quad (2.3)$$

where  $\delta_j$ ,  $j = 1, \dots, \mu$ , form a marked set of vanishing cycles;  $\{\omega_i | i = 1, \dots, \mu\}$  is a standard set of forms (2.2).

When we want to specify dependence on  $H$ , we write  $\mathbb{I}(\hat{t}, H)$  instead of  $\mathbb{I}(\hat{t})$ .

## 2.4 Upper estimates of integrals

Denote by  $|\lambda|$  the length of a curve  $\lambda$ , and by  $U^\varepsilon(A)$  the  $\varepsilon$ -neighborhood of a set  $A$ .

The main result of the quantitative algebraic geometry that we need is the following

**2.10 Theorem** *Let  $\delta_j$  be a vanishing cycle from a marked set, see Definition 2.5, corresponding to a curve  $\alpha_j$ ,  $|\alpha_j| \leq 9$  (recall that  $|t_0| \leq 3$ ). Let  $\lambda \subset B$  be a curve starting at  $t_0$  (denote  $t$  its end) such that*

$$|\lambda| \leq 36n^2 + 1, \quad |t| \leq 4. \quad (2.4)$$

*Let the curve  $\alpha_j \cap U^\varepsilon(a_j)$  be a connected arc of  $\alpha_j$ , and the curves  $\alpha_j \setminus U^\varepsilon(a_j)$  and  $\lambda$  have an empty intersection with  $\varepsilon$ -neighborhoods of the critical values  $a_k$ , where  $\varepsilon = \nu/2$ ,  $\nu$  is from (1.2). Let  $\omega$  be a form (2.2),  $\hat{t} \in W$  corresponds to  $[\lambda]$ , and  $\delta_j(\hat{t})$  be the extension of  $\delta_j$  to  $\hat{t}$ . Then*

$$|I_{\delta_j(\hat{t})}\omega| < 2^{\frac{2600n^{16}}{c'(H)}} (c'(H))^{-28n^4} := M_0 \quad (2.5)$$

This result is based on Theorem C from 1.5. Both results are proved in the forthcoming paper [3].

We have to give an upper bound of the integral not over a vanishing cycle, but over a real oval. The following Lemma shows that the real oval is always a linear combination of some (at most  $\mu$ ) vanishing cycles with coefficients  $\pm 1$ .

**2.11 Lemma (Geometric lemma).** *Let  $H$  be a real ultra-Morse polynomial and  $\gamma$  be a real oval of  $H$ . Let  $H|_\gamma = t_0$ . Consider the critical values of  $H$  that correspond to the critical points located inside  $\gamma$  in the real plane. Let  $\alpha_j$ ,  $j = 1, \dots, s$ , be nonintersecting and nonself-intersecting paths that connect  $t_0$  with these critical values and satisfy assumption (2.1); we may change the numeration of critical points to get the first  $s$  ones inside  $\gamma$ . Moreover, suppose that all these paths belong to the upper halfplane and no open domain bounded by a path  $\alpha_j$  and a real segment (connecting the endpoints of  $\alpha_j$ ) contains any critical value of  $H$  (see Fig.2a,b). Let  $\delta_j$  be the vanishing cycles that correspond to the paths  $\alpha_j$ . Then*

$$[\gamma] = \sum_1^s \varepsilon_j \delta_j, \quad \text{where } \varepsilon_j = \pm 1. \quad (2.6)$$

The authors believe that Lemma 2.11 is well known to specialists, but they did not find it in literature. Its proof is given in 3.5.

**2.12 Corollary** *The integral (1.1) constructed for the real oval  $\gamma = \gamma(t_0)$  satisfies the upper estimate:*

$$\left| \int_{\gamma(\hat{t})} \omega \right| \leq n^2 \max_{j=1, \dots, s} \left| \int_{\delta_j(\hat{t})} \omega \right|.$$

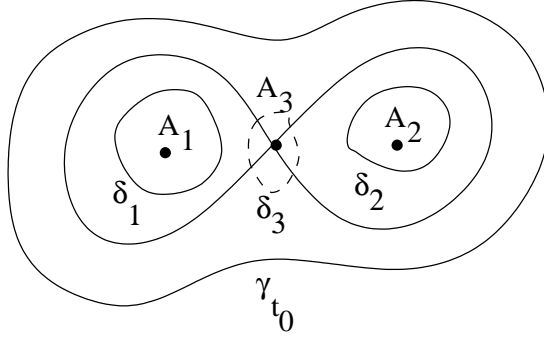
**2.13 Corollary** *In the condition of the previous Theorem let  $H$  be a real polynomial,  $\gamma(\hat{t})$  be the extension to  $\hat{t}$  of a real oval,*

$$\omega = \sum_{k+l \leq n-1} a_{kl} x^k y^l dx. \quad (2.7)$$

*Then*

$$|I_{\gamma(\hat{t})}\omega| \leq n^4 M_0 \max_{k+l \leq n-1} |a_{kl}|. \quad (2.8)$$

a) The cycle  $\gamma_{t_0}$  and local vanishing cycles  $\delta_j = \delta_j(t_j)$ ; the points  $t_j$  close to  $a_j$  are marked at Fig.b).



b) The paths for the extension of the local vanishing cycles  $\delta_j(t)$ .

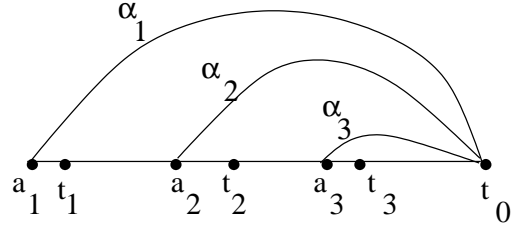


Figure 2:

## 2.5 Determinant of periods

The determinant of the matrix of periods (2.3) is called the *determinant of periods*. It appears that this determinant is single-valued on  $B$ , thus depending not on a point of the universal cover  $W$ , but rather on the projection of this point to  $B$ . Let

$$\Delta(t) = \det \mathbb{I}(\hat{t}), \quad t = \pi \hat{t}.$$

The single-valuedness of the main determinant follows from the Picard-Lefschetz theorem. Indeed, a circuit around one critical value adds the multiple of the correspondent column to some other columns of the matrix of periods. Thus the determinant remains unchanged.

When we want to specify the dependence of the main determinant on  $H$ , we write  $\Delta(t)$ . This function is polynomial in  $t$ , and an algebraic function in the coefficients of  $H$ . The formula for the main determinant (with  $\omega_i$  of appropriate degrees) with a sketch of the proof was claimed by A.Varchenko [22]; this formula is given up to a constant factor not precisely determined. The complete answer (under the same assumption on the degrees of  $\omega_i$ ) is obtained by the first author (A.Glutsyuk, [2]). Moreover, the following lower estimate holds:

**2.14 Theorem** *For any normalized ultra-Morse polynomial  $H$ , the tuple  $\Omega$  of standard forms (2.2) may be so chosen that for any  $t \in \mathbb{C}$  lying outside the  $\nu = \frac{c''}{4n^2}$ -neighborhoods of the critical values of  $H$  the following lower estimate holds:*

$$|\Delta(t, H)| \geq (c'(H))^{6n^3} (c''(H))^{n^2} n^{-62n^3} := \Delta_0 \quad (2.9)$$

This result is proved in [3] with the use of the explicit formula for the Main Determinant mentioned before, and results of the quantitative algebraic geometry.

## 2.6 Construction of the set $K$

We can now pass to the construction of the set  $K$  mentioned in the Main Lemma. We first construct a smaller set  $K'$ .

**2.15 Lemma (Construction lemma).** *Let  $\gamma \subset S_{t_0}$  be a real oval. There exists a set of regular paths  $\alpha_j$ ,  $j = 1, \dots, \mu$ , see Definition 2.4, such that:*

$$|\alpha_j| \leq 9,$$

*the paths  $\alpha_j$  are not pairwise and self intersected;*

*and there exists a path connected set  $K' \subset W$ ,  $t_0 \in K'$ ,  $\pi K' \subset D_3$ , such that:*

*for any cycle  $\delta_j \in H_1(S_{t_0}, \mathbb{Z})$  vanishing along  $\alpha_j$  there exist two points  $\tau_1, \tau_2 \in K' \cap \pi^{-1}(t_0)$  such that*

$$[\gamma(\tau_1)] - [\gamma(\tau_2)] = l_j[\delta_j], \quad l_j \in \mathbb{Z} \setminus 0. \quad (2.10)$$

*Moreover,*

$$\text{diam}_{\text{int}} K' < 19n^2, \quad (2.11)$$

*and  $\pi K'$  is disjoint from  $\nu$ -neighborhoods of the critical values  $a_j$ ,  $j = 1, \dots, \mu$ .*

**2.16 Lemma (Construction lemma for vanishing cycles).** *Construction lemma holds true if  $\gamma \subset S_{t_0}$  is replaced in its statement by any vanishing cycle  $\delta_l = \delta_l(t_0)$  from an arbitrary marked set of vanishing cycles, and  $W$  is replaced by  $W_l$ . In the conclusion, (2.10) should be replaced by*

$$[\delta_l(\tau_1)] - [\delta_l(\tau_2)] = l_j[\delta_j], \quad \text{for } j \neq l, \quad [\delta_l(\hat{t})] = [\delta_l] \text{ for } \hat{t} = t_0, \quad l_j \in \mathbb{Z} \setminus 0.$$

Both lemmas are proved in 2.9. In what follows we deduce the Main Lemma from Lemma 2.15 and Theorems 2.10, 2.14.

**2.17 Corollary (of Lemma 2.15).** *For any form  $\omega$  (not necessarily of type (2.2)) and any marked set of vanishing cycles consider the vector function*

$$\mathbb{I}_\omega : W \rightarrow \mathbb{C}^\mu, \quad \hat{t} \mapsto \left( \int_{\delta_1(\hat{t})} \omega, \dots, \int_{\delta_\mu(\hat{t})} \omega \right). \quad (2.12)$$

*Let  $\|\cdot\|$  denote the Euclidean length in  $\mathbb{C}^\mu$ . Then*

$$m_0 := \max_{\hat{t} \in K' \cap \pi^{-1}(t_0)} |I(\hat{t})| \geq \frac{1}{2n} \|\mathbb{I}_\omega(t_0)\|. \quad (2.13)$$

**Proof** Let us take  $j$  so that

$$\max_i \left| \int_{\delta_i(t_0)} \omega \right| = \left| \int_{\delta_j(t_0)} \omega \right|.$$

Then

$$\left| \int_{\delta_j(t_0)} \omega \right| \geq \frac{1}{n} \|I_\omega(t_0)\| \quad (2.14)$$

By Lemma 2.15, there exist  $\tau_1, \tau_2$  such that

$$I(\tau_1) - I(\tau_2) = l \int_{\delta_j(t_0)} \omega, \quad l \in \mathbb{Z} \setminus 0.$$



Hence, at least one of the integrals in the left hand side, say,  $I(\tau_l)$ ,  $l \in \{1, 2\}$ , admits a lower estimate:

$$|I(\tau_l)| \geq \frac{1}{2} \left| \int_{\delta_j(t_0)} \omega \right|. \quad (2.15)$$

Together with (2.14) this proves the corollary.  $\square$

Let us now take

$$K = K' \cup \Sigma, \Sigma = \sigma(t_0) \cup L^\pm(t_0) \cup R^\pm(t_0). \quad (2.16)$$

In the following section we will check that this  $K$  satisfies the requirements of the Main Lemma.

## 2.7 Proof of the Main Lemma

Let us take  $K$  as in (2.16). Let  $\nu$  be the same as in (1.2). Let  $U$  be the smallest simply connected set that contains the  $\varepsilon$ -neighborhood of  $K$ ,  $\varepsilon = \nu/2$ . Then (1.11) follows from (2.11), (2.16). The last statement of Lemma 2.15 implies (1.12).

Let us now check (1.13), that is, estimate from above the Bernstein index  $B_{K,U}(I)$ .

By Theorem 2.10 and (1.11), all the elements of the matrix  $\mathbb{I}(t_0)$  admit an upper bound:

$$|I_{ij}(t_0)| < M_0.$$

Fix a form  $\omega_0 = A_n dx + B_n dy$ . There exists another form  $\omega$  of the type (2.7) such that the form  $\omega - \omega_0$  is exact. Let  $\omega_i = yx^{k_0}y^{l_0}dx$  be such that  $|a_{k_0l_0}| = \max_{k+l \leq n-1} |a_{kl}|$  in (2.7). Without loss of generality we set  $a_{k_0l_0} = 1$ . Let us now replace the  $i$ th row of the matrix  $\mathbb{I}$  by the vector  $\mathbb{I}_\omega$ . This transformation is equivalent to adding to the  $i$ -th line linear combination of other lines, so the determinant  $\Delta(t_0)$  remains unchanged. All the elements in all other rows are estimated from above by  $M_0$ . Hence, all the vector-rows except for the  $i$ th one have the length at most  $nM_0$ . By (2.13), the  $i$ th row has the length at most  $2nm_0$ . Hence,

$$\Delta_0 \leq 2m_0 M_0^{\mu-1} n^\mu, \quad \mu = n^2$$

where  $\Delta_0$  and  $M_0$  are from (2.9) and (2.5) respectively. Therefore,

$$\log m \geq \log m_0 \geq \log \Delta_0 - (\mu - 1) \log M_0 - \mu \log n - \log 2. \quad (2.17)$$

On the other hand, by (2.8),

$$\max_U |I| \leq n^4 M_0.$$

Hence,

$$B_{K,U}(I) \leq (\mu + 4) \log n + \mu \log M_0 - \log \Delta_0 + \log 2.$$

Now, elementary estimates imply (1.14). This proves the Main Lemma.

## 2.8 Modified Main Lemma and zeros of integrals over (complex) vanishing cycles

**Proof of the Modified Main Lemma.** The arguments of the previous section work almost verbatim. The previous Corollary for  $\gamma$  replaced by  $\delta_l$  is stated and proved in the same way.

Let  $K'$  be the same as in Lemma 2.16. Instead of (2.16), let

$$K = K' \cup \alpha_l \cup \overline{D}_\nu(a_l).$$

Let  $U$  be the smallest simply connected set that contains the  $\varepsilon$ -neighborhood of  $K$ .

By Theorem 2.10,

$$\max_{\overline{V}} |I_l| \leq M_0, \text{ where } V = U \setminus D_\nu(a_l).$$

But  $I_l$  is holomorphic in  $D_\nu(a_l)$ . Hence, by the maximum modulus principle, the previous inequality holds in  $U$  instead of  $V$ . After that, the rest of the arguments of the previous section work. This proves the Modified Main Lemma.  $\square$

**2.18 Theorem** *The number of zeros of the integral  $I_l$  in the disk  $D_\nu(a_l)$  satisfies the inequality:*

$$\#\{\hat{t} \in D_\nu(a_l) | I_l(\hat{t}) = 0\} \leq (1 - \log c'(H))A^{578}. \quad (2.18)$$

The proof is the same as for Theorem A1, section 1.7.

## 2.9 Proof of the Construction Lemmas

**Proof of Lemma 2.15.**

**2.19 Definition** A loop  $\lambda_j$  is associated to a regular path  $\alpha_j$  if

$$\lambda_j = \tilde{\alpha}'_j \partial D_\nu(a_j) (\tilde{\alpha}'_j)^{-1},$$

where  $\tilde{\alpha}'_j = \alpha_j \setminus D_\nu(a_j)$ ,  $\nu = \frac{c''}{4n^2}$ ,  $\partial D_\nu(a_j)$  is positively oriented (we suppose that  $\tilde{\alpha}'_j$  is connected).

Let  $\alpha_1, \dots, \alpha_\mu$  be the same as in Definition 2.5. The set  $K'$  we are looking for will be the union of appropriate  $n^2$  liftings of the loops  $\lambda_j$  (one lifting for each  $\lambda_j$ ) associated with  $\alpha_j$  to the Riemann surface  $W$ . In what follows, the choice of the curves  $\alpha_j$  will be specified.

We prove Lemma 2.15 in four steps. The set  $K'$  is constructed in the first three steps. In the fourth step we check that the resulting set has the required properties.

Step 1: special path set. Denote by  $\alpha'_j$  the segment  $[t_0, a_j]$  oriented from  $t_0$ . Fix  $j$  and suppose that  $\alpha'_j$  contains critical values of  $H$  different from  $a_j$ ; denote the set of these values by  $A$ . For any  $a_i \in A$  replace the diameter  $\alpha'_j \cap D_\nu(a_i)$  by a semicircle.

If  $a_j$  is real, then this semicircle is chosen in the upper half-plane. In general, for any fixed line  $\theta$  passing through  $t_0$  and some critical values the previous semicircles corresponding to all the pairs  $a_i, a_j$  in  $\theta$  as above are chosen to be on one and the same side from  $\theta$ . If  $\alpha'_j$  intersects a disc  $D_\nu(a_s)$  but does not contain  $a_s$ , replace the chord  $\alpha'_j \cap D_\nu(a_s)$  by the

smallest arc of the circle  $\partial D_\nu(a_s)$ . The path thus constructed will be denoted by  $\alpha_j$ . Recall that  $t_0 \in \bar{D}_3$ ,  $a_j \in D_2$ . Therefore, the length of any segment  $\alpha'_j$  is less than 5. Hence,

$$|\alpha_j| \leq \frac{5\pi}{2} < 9. \quad (2.19)$$

Each path  $\alpha_j$  is nonself-intersected by construction and is contained in  $D_3$  (except may be for  $t_0$ ). One can achieve that the paths  $\alpha_j$  be disjoint outside  $t_0$  by applying to them arbitrarily small deformation preserving the previous inequality and inclusion.

Step 2: special loop set. For any  $j$  denote by  $\lambda_j$  the loop associated to  $\alpha_j$  in the sense of Definition 2.19. By construction,  $\lambda_j \subset \bar{D}_3$ . We have

$$|\lambda_j| \leq 2|\alpha_j| + |\partial D_\nu(a_j)| < 19.$$

Step 3: construction of  $K'$ . Denote by  $G$  the intersection graph of  $\gamma_{t_0}$  and all the vanishing cycles  $\delta_i$  (along the previously constructed paths  $\alpha_i$ ). *This graph is connected.* This follows from the two lemmas below.

**2.20 Lemma** *The intersection graph of the marked set of vanishing cycles is connected. The set itself forms a basis in the group  $H_1(S_{t_0}, \mathbb{Z})$ .*

(Recall the definition of the intersection graph: its vertices are identified with the cycles; two of them are connected by an arc, if and only if the corresponding intersection index is nonzero.)

Lemma 2.20 is implied by the following statements from [1]: theorem 1 in 2.1 and theorem 3 in 3.2.

**2.21 Lemma** *Consider a maximal family of real ovals that contains  $\gamma_{t_0}$ . The union of the ovals of the family forms an open domain. The boundary of this domain consists of one or two connected components. Any of these components belongs to a critical level of  $H$  and contains a unique critical point. Fix any of these critical points and denote by  $\delta$  the corresponding local vanishing cycle. Then the cycle  $\delta$  may be extended to a cycle  $\delta(t_0)$  that belongs to a marked set of vanishing cycles constructed above. Moreover,*

$$(\delta(t_0), \gamma_{t_0}) \neq 0, \text{ more precisely, it is equal to } \pm 1, \pm 2.$$

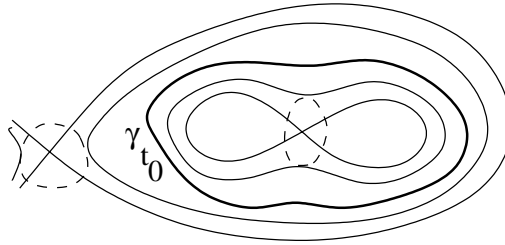
The proof of this lemma is written between the lines of [4], pp 12,13. It is illustrated by Fig.3.

Let us define a metric on the set of the vertices of the graph  $G$ . Suppose that each edge of  $G$  has length 1. Then the distance  $D_G$  between any two vertices of  $G$  is well defined as the length of the shortest path in  $G$  that connects the vertices. For any  $r \in \mathbb{N}$  let

$$S_r = \{\delta_j \mid D_G(\gamma_{t_0}, \delta_j) = r\}.$$

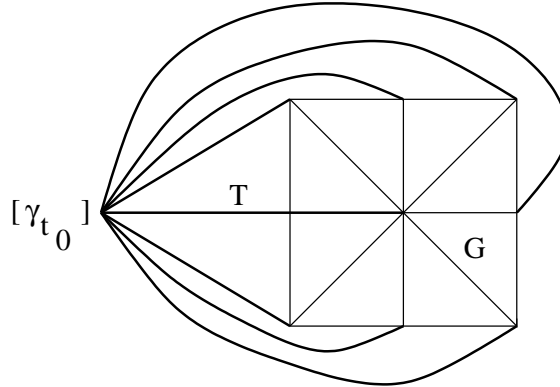
Let  $T$  be a maximal tree in  $G$  with the root  $[\gamma_{t_0}]$  such that the distance in  $T$  (defined as  $D_G$  but with paths in  $T$ ) of any vertex to the root  $[\gamma_{t_0}]$  coincides with  $D_G$  (see Fig.4, where the tree  $T$  is marked by bold curves.)

For any vanishing cycle  $\delta_j(t_0)$  let  $L_j$  be the branch of the tree  $T$  from  $[\gamma_{t_0}]$  to  $\delta_j(t_0)$ . Let  $[\gamma_{t_0}], \delta_{j_1}(t_0), \dots, \delta_{j_r}(t_0) = \delta_j(t_0)$  be its vertices ordered from the beginning to the end of



Local vanishing cycles having nonzero intersection index with the ovals of the family that contains  $\gamma_{t_0}$

Figure 3:



Intersection graph and the tree T

Figure 4:

the branch. By definition, the intersection index of any cycle in this sequence with its two neighbors is nonzero, and that of any two nonneighbor cycles is zero. Let us call this *the regularity property* of  $L_j$ .

The set  $K' \subset W$  we are looking for is the image of the tree  $T$  in  $W$  under a continuous map  $\phi : T \rightarrow W$ . This map is defined by induction in  $r$  as follows. It suffices to define  $\phi|_{L_j}$  for any  $\delta_j$ .

Base of induction:  $r = 0$ . The cycle  $\gamma_{t_0}$  is mapped to  $t_0$ .

Induction step. Suppose that the cycle  $\delta' = \delta_{j_{r-1}}$  is mapped to  $\tau_1 \in \pi^{-1}(t_0)$ :  $\phi(\delta') = \tau_1$ . Let us lift the loop  $\lambda = \lambda_{j_r}$  to  $W$  as a covering curve  $\tilde{\lambda}$  over  $\lambda$  with the starting point  $\tau_1$ . Let  $\delta = \delta_{j_r}$ ,  $\tau_2 \in \pi^{-1}(t_0)$  be the endpoint of  $\tilde{\lambda}$ . This induces a map of the edge  $[\delta', \delta]$  to  $\tilde{\lambda}$ . This map defines the extension of  $\phi$  to the edge  $[\delta', \delta]$ . The induction step is over.

Step 4: properties of the set  $K'$ . The set  $K'$  is a curvilinear tree and thus, path connected.

Its intrinsic diameter admits the upper estimate

$$\text{diam}_{\text{int}} K' \leq n^2 \max_j |\lambda_j| < 19n^2.$$

The set  $K'$  is projected to the loops  $\lambda_j$ , which lie in  $\bar{D}_3$  and are disjoint from the  $\nu$ -neighborhoods of the critical values by definition. Hence, the same is true for  $\pi(K')$ .

For any cycle  $\delta = \delta_j(t_0)$  vanishing along the path  $\alpha_j$  from the special path set, see Step 1, let  $L$  be the edge of the tree  $T$  with the endpoint  $\delta$ . Let  $\delta'$  be the initial point of  $L$ . Let  $\tau_1 = \phi(\delta')$ ,  $\tau_2 = \phi(\delta)$ . Then (2.10) holds by the Picard-Lefschetz theorem. In more details, let  $L_j$  and  $\delta_{j_m}(t_0)$  be the same, as in Step 3. Then

$$\gamma_{\tau_2} = \gamma_{t_0} + \sum_{m=1}^r l_m \delta_{j_m}(t_0), \quad l_m \in \mathbb{Z} \setminus 0, \quad (2.20)$$

$$\gamma_{\tau_1} = \gamma_{t_0} + \sum_{m=1}^{r-1} l_m \delta_{j_m}(t_0), \quad l_m \in \mathbb{Z} \setminus 0. \quad (2.21)$$

Let us prove (2.20) by induction in  $r$  taking (2.21) as the induction hypothesis. Equality (2.21) implies (2.20) by Picard-Lefschetz theorem [1] and the regularity property of  $L_j$ , see Step 3. On the other hand, (2.20) and (2.21) imply (2.10). Lemma 2.15 is proved.  $\square$

Lemma 2.16 is proved in the same way with the following minor changes:  $G$  is now the intersection graph of the marked set of vanishing cycles considered, and in the lifting process,  $W$  should be replaced by  $W_l$ .

### 3 Number of zeros of abelian integrals in complex domains distant from critical values

In this section we prove the first part of Theorem B, namely,

#### 3.1 Upper estimates in Euclidean and Poincaré disks

Theorem B1 mentioned in 1.5 is stated below.

All through this section notations of section 2.1 hold. Moreover,  $\delta_1, \dots, \delta_\mu$  is a marked set of vanishing cycles on  $S_{t_0}$ ,  $K \subset W$  is a compact set from the Main Lemma, see 1.7.

**3.1 Theorem** *Fix a normalized polynomial  $H$ . Let  $\hat{t} \in W$  be a point represented by a curve  $\lambda \subset B$ . Let  $\delta_j$  be a vanishing cycle from a marked set corresponding to a curve  $\alpha_j$ . Let  $\alpha = \lambda^{-1}\alpha_j$ . Let  $0 < \beta < 1$ . Suppose that  $\alpha \cap D_\beta(a_j)$  is a connected arc of the path  $\alpha$ , and  $\alpha$  avoids the  $\beta$ -neighborhoods of the critical values distinct from  $a_j$  of the polynomial  $H$ . Then for any 1-form  $\omega$  of type (2.2)*

$$\left| \int_{\delta_j(\hat{t})} \omega \right| < 2^{-2n} M_1, \quad M_1 = 2^{10n^{12} \frac{|\alpha|+5}{\beta}} (c'(H))^{-28n^4} \quad (3.1)$$

Theorem 3.1 is proved in [3]. It is used in the estimate of the number of zeros in Euclidean disc. The following upper bound (Theorem 3.3 also proved in [3]) of integrals is used to prove an upper bound of the number of zeros in Poincaré disc that is exponential in the radius of the disc.

**3.2 Remark** One can estimate the number of zeros in Poincaré disc by using Theorem 3.1 instead of Theorem 3.3 (see the proof for Euclidean disc below). But the upper bound of the number of zeros obtained in this way is double exponential in the radius.

Denote by  $V_\gamma f$  the variation of the argument of the function  $f$  along an oriented curve  $\gamma$ .

**3.3 Theorem** [3]. *Let  $H$  be a normalized ultra-Morse polynomial of degree  $n + 1 \geq 3$ . Let  $\hat{t} \in W$  be a point represented by a curve  $\lambda \subset B$ . Let  $\delta$  be a vanishing cycle from the marked setp;  $\delta$  corresponds to a curve  $\alpha_j$ ,  $\alpha = \lambda^{-1}\alpha_j : [0, 1] \rightarrow B$ . Let  $0 < \beta \leq \nu = \frac{c'(H)}{4n^2}$ ,*

$$t' = \alpha(0) = \pi(\hat{t}), \quad a = \alpha(1) = \alpha_j(1), \quad \tau' = \min\{\tau \in [0, 1] \mid \alpha(\tau, 1] \subset D_\beta(a)\}, \quad \hat{\alpha} = \alpha \setminus \alpha(\tau', 1],$$

$$\tilde{\alpha} = \alpha \cap (\overline{D}_3 \setminus \cup_i D_\beta(a_i)), \quad (3.2)$$

$$V = V_{\alpha, \beta} = \beta \sum_i V_{\hat{\alpha} \cap D_\beta(a_i)}(t - a_i) + 3V_{\hat{\alpha} \setminus \overline{D}_3} t.$$

Let  $\delta \in H_1(S_{t'}, \mathbb{Z})$  be the cycle vanishing along  $\alpha$ . Let  $\omega$  be a monomial 1- form of degree at most  $2n - 1$  with unit coefficient. Then

$$|I_\delta(\hat{t})| \leq 2^{-2n} M_2, \quad M_2 = 2^{20n^{12} \frac{|\tilde{\alpha}| + V + 5}{\beta}} (c'(H))^{-28n^4} \max\{1, (\frac{|t'|}{5})^2\}. \quad (3.3)$$

Let  $DE_{R, \beta}$  be an Euclidean disk in  $W$  with  $\beta$ - neighborhoods of critical values deleted. More precisely,  $DE_{R, \beta}$  is the set of all those  $\hat{t} \in W$  that may be represented by a curve  $\lambda$ , whose length is no greater than  $R$ , provided that  $\lambda$  avoids  $\beta$ -neighborhoods of critical values.

**3.4 Theorem** *Let  $H$  be a normalized complex ultra-Morse polynomial of degree  $n + 1 \geq 3$ ,  $\omega$  be arbitrary 1- form of degree at most  $n$ . Then the number of zeros of integral (1.1), which is an analytic extension of an integral over real ovals or over marked vanishing cycles of a normalized polynomial  $H$ , is estimated from above as follows:*

$$\#\{\hat{t} \in DE_R \mid I(t) = 0\} < (1 - \log c'(H)) e^{\frac{9R}{\beta}} \quad (3.4)$$

provided that

$$R \geq 36n^2, \quad \beta \leq \nu/2 \quad (3.5)$$

The following statement is an analogue of Theorem 3.4 for Euclidean metric replaced by the Poincaré one.

**3.5 Theorem (Theorem B1).** *In the assumptions of Theorem 3.4 the number of zeros of integral (1.1) over real ovals or over marked vanishing cycles of a normalized polynomial  $H$  is estimated as follows:*

$$\#\{\hat{t} \in DP_R \mid I(\hat{t}) = 0\} < (1 - \log c'(H)) e^{7R}, \quad (3.6)$$

provided that

$$R \geq \frac{288n^4}{c''(H)}. \quad (3.7)$$

Recall that  $DP_R$  is the disk in the Poincaré metric of  $W$  of radius  $R$  centered at the base point  $t_0$ .

Theorem B1 forms the first part of Theorem B. The second part of Theorem B, Theorem B2, is presented in Section 4. Theorems B1, B2 imply Theorem B.

### 3.2 Idea of the proof

Theorems 3.4 and B1 are proved as Theorem A1, making use of Growth-and-Zeros Theorem. The set  $K$ , both from the Main Lemma and from the Modified Main Lemma, belongs to  $DE_R$  by (3.5) and to  $DP_R$  by (3.7), see (1.11) and Remark 1.15.

Thus we have the main ingredient in the estimate of the Bernstein index, namely, the lower bound for  $m$ , see (2.17).

An upper estimate for the integral over a vanishing cycle is provided by Theorem 3.1. Yet there is a gap to be filled when we wish to replace a vanishing cycle by a real oval. This is done by the following corollary of the Geometric Lemma 2.11.

**3.6 Corollary** *In the assumptions of Theorem 3.1 and any real oval  $\gamma$  of  $H$ ,*

$$\left| \int_{\gamma} \omega \right| < \frac{M_1}{n^2}, \quad (3.8)$$

see (3.1).

Now everything is ready for the application of the Growth-and Zeros theorem.

### 3.3 Number of zeros in a Euclidean disk

**Proof of Theorem 3.4.** Denote the closure of the domain  $DE_R$  by  $K$ . Let  $\varepsilon' = \beta/2$ ,  $U$  be the smallest simply connected domain in  $W$  that contains the  $\varepsilon'$ -neighborhood of  $K$ . Then

$$D := \text{diam}_{\text{int}} K \leq 2R, \quad \pi\text{-gap}(K, \partial U) = \varepsilon'.$$

Hence,

$$e^{\frac{2D}{\varepsilon}} \leq e^{\frac{8R}{\beta}}.$$

This is the main factor in the estimate (3.4).

Let us now estimate from above the Bernstein index  $B = B_{K,U}(I)$ . Let  $K'$  be the set from Lemma 2.15 (case of real oval) or Lemma 2.16 (case of vanishing cycle). Let

$$m = \max_K |I|.$$

One has  $K' \subset K$  by (2.11) and (3.5). Therefore,  $\log m \geq \log m_0$ , where  $\log m_0$  is from (2.13). On the other hand, let

$$M = \max_U |I|.$$

As in the proof of the Main Lemma, we assume (without loss of generality) that  $\omega$  is of the type (2.7) with  $\max |a_{kl}| = 1$ . Then by Corollary 3.6 (case of real cycle) or Theorem 3.1 (case of vanishing cycle), one has

$$M \leq M_1,$$

$M_1$  is from (3.1). Then

$$B_{K,U}(I) \leq \log M_1 - \log m_0.$$

Inequalities (2.17), (3.5) together with elementary estimates imply that

$$\log M_1 - \log m_0 < (1 - \log c'(H))e^{\frac{R}{\beta}}.$$

Together with Growth-and-Zeros theorem, this completes the proof of Theorem 3.4.  $\square$

### 3.4 Number of zeros in a disk in the Poincaré metric

The proof of Theorem B1 is carried on by application of version (1.8) of the Growth-and-Zeros Theorem to the sets

$$K_R = \overline{DP_R}, \quad U_R = DP_{R+1} :$$

$$\#\{\hat{t} \in \overline{DP_R} \mid I(\hat{t}) = 0\} \leq B_{K_R, U_R} e^{\rho_R}, \quad \rho_R = \text{diam}_{PU_R} K_R. \quad (3.9)$$

The right-hand side of the latter inequality is estimated below.

The set  $K$  from the Main Lemma is contained in  $K_R$  (this follows from (1.16)), and as before, this yields immediately lower bound of  $m$ . The principal part of the proof of Theorem B1 is the upper bound of the integral on the set  $U_R$ :

$$\max_{\overline{U_R}} |I(\hat{t})| < M(R), \quad \log M(R) = (1 - \log c'(H))e^{1.2R}. \quad (3.10)$$

To prove it, we use Theorem 3.3. Namely, given a  $\hat{t} \in U_R$ , consider the path  $\hat{\lambda}$  that is the geodesic from  $t_0$  to  $\hat{t}$  in the Poincaré metric of  $W$  (we put  $\lambda = \pi(\hat{\lambda})$ ) and the path

$$\alpha = \lambda^{-1} \alpha_i \text{ from } t = \pi(\hat{t}) \text{ to } a_i. \text{ Put } \beta = \nu = \frac{c''}{4n^2}.$$

We have to estimate from above the value  $M_2$  from Theorem 3.3, in particular, to estimate from above the module  $|t|$  and the linear combination  $V_{\alpha, \nu}$  of variations. To do this, we prove the following upper bound of the radius of the closed Euclidean disc containing  $\overline{DP_{R+1}} \supset \lambda$  and lower bound of the gap between  $DP_{R+1}$  and the critical values of  $H$ :

$$\pi(\overline{DP_{R+1}}) \subset D_{M_R}, \quad \text{where } \log M_R = 6e^R \log R, \quad (3.11)$$

$$\text{dist}(\overline{\pi(DP_{R+1})}, a_i) > \beta_R, \quad \text{where } \beta_R = M_R^{-1}. \quad (3.12)$$

Using the two latter inequalities we show that

$$V_{\alpha, \nu} < 37e^R R \log R. \quad (3.13)$$



The proofs of (3.11)-(3.13) and upper bound of  $|\tilde{\alpha}|$  ( $\tilde{\alpha}$  is defined in (3.2)) are based on the following lower bounds of the Poincaré metric. Given a domain  $G \subset \mathbb{C}$ ,  $\#(\mathbb{C} \setminus G) > 1$ , denote by  $P(G)$  the ratio of the Poincaré metric of  $G$  to the Euclidean one;  $P(G)$  is a function in  $t \in G$ .

**Inequality** (follows from theorem 2.17 in [20]). For any distinct  $a, b \in \mathbb{C}$  one has

$$P(\mathbb{C} \setminus \{a, b\})(t) > [\min_{c=a,b} |t - c| (\min_{c=a,b} |\log |\frac{t-c}{a-b}|| + 5)]^{-1} \quad (3.14)$$

**3.7 Corollary** *Let  $H$  be a balanced polynomial,  $B$  be the complement of  $\mathbb{C}$  to its critical values. Then*

$$P(B)(t) > [|t - a| (|\log |t - a|| + C)]^{-1}, C = 2 \log n - \log c''(H) + 5, \text{ for any critical value } a. \quad (3.15)$$

The Corollary follows from the previous Inequality and monotonicity of the Poincaré metric. **Proof of (3.10).** Let  $\alpha_i$  be a path from  $t_0$  to  $a = a_i$  from a marked path set,  $|\alpha_i| \leq 9$ ,  $\alpha = \lambda^{-1} \alpha_i$ . As in the previous Subsection (without loss of generality we consider that the form  $\omega$  has the type (2.7) with  $\max |a_{kl}| = 1$ ), one has

$$|I(\hat{t})| < n^4 (2^{-2n} M_2) \leq M_2, \quad (3.16)$$

where  $M_2$  is the same as in (3.3) (recall that  $\beta = \nu$ ). Let us estimate  $M_2$ : we show that

$$\log M_2 < (1 - \log c') R^6 e^R. \quad (3.17)$$

By elementary inequalities, the latter right-hand side is less than  $\log M(R)$ . This together with (3.16) implies (3.10).

The linear combination  $V$  of variations and  $|t'| = |t|$  are estimated by inequalities (3.11) and (3.13) respectively (proved below). Let us estimate the length of  $\tilde{\alpha}$ : we show that

$$|\tilde{\alpha}| < 12R \log R. \quad (3.18)$$

By definition, the curve  $\tilde{\alpha}$  consists of the arcs of paths  $\lambda$  and  $\alpha_i$  lying in  $\overline{D}_3 \setminus \cup_i D_\nu(a_i)$ . Those contained in  $\alpha_i$  have total length less than 9, since  $|\alpha_i| \leq 9$ . Those contained in  $\lambda$  have total length no greater than

$$|\lambda|_P M_3, \quad M_3 = (\min_{\tilde{\alpha}} P(B))^{-1}, \quad |\lambda|_P \text{ is the Poincaré length, thus, } |\lambda|_P \leq R + 1.$$

Let us estimate  $M_3$ . Recall that the curve  $\tilde{\alpha}$ , where the minimum in  $M_3$  is taken, lies in  $\overline{D}_3$  and its gap from the critical values is no less than  $\nu$ . This together with (3.15) and the inequality  $|a| \leq 2$  implies

$$M_3 \leq \max_{|t| \leq 3} |t - a| (-\log \nu + C) \leq 5(-\log \nu + C).$$

Inequality (3.7) together with elementary inequalities implies that

$$C < \log R, \quad -\log \nu < \log R. \quad (3.19)$$

Therefore,  $M_3 < 10 \log R$ . This together with the previous discussion and (3.7) implies that

$$|\tilde{\alpha}| < 9 + |\lambda|_P 10 \log R < 11(R+1) \log R < 12R \log R.$$

This proves (3.18). Substituting itself, (3.13) and the inequality  $|t'| = |t| < M_R$  (which follows from (3.11)) to the expression (3.3) of  $M_2$  we get

$$\log M_2 < 20n^{12} \frac{12R \log R + 37e^R R \log R + 5}{\nu} - 28n^4 \log c' + 2 \log \frac{M_R}{5}.$$

By elementary inequalities and (3.7), the latter right-hand side is less than

$$e^R R^5 \log R - R \log c' + 12e^R \log R < (1 - \log c') R^6 e^R.$$

This proves (3.17) and (3.10). □

**Proof of Theorem B1.** One has

$$\rho_R < 5R. \tag{3.20}$$

This follows from the fact that the diameter of  $K_R = DP_R$  in the Poincaré metric of  $W$  is equal to  $2R$  (by definition), and the inequality

$$\frac{PU_R}{PW}|_{K_R} \leq \frac{e+1}{e-1} < \frac{5}{2}.$$

The latter inequality is a particular case of the following more general statement.

**3.8 Proposition** *Let  $W$  be a hyperbolic Riemann surface,  $U \subset W$  be a domain,  $K \Subset U$  be a compact set. Let  $\text{dist}_{PW}(K, \partial U) \geq \sigma > 0$ . Then*

$$\frac{PU}{PW}|_K \leq \frac{e^\sigma + 1}{e^\sigma - 1}.$$

**Proof** By monotonicity of the Poincaré metric as a function of domain, it suffices to prove the Proposition in the case, when  $W = D_1$ ,  $K = \{0\}$ ,  $U$  is the Poincaré disc of radius  $\sigma$  centered at 0: in this case we prove the equality. Indeed, let  $r$  be the Euclidean radius of the latter disc. By definition and conformal invariance of the Poincaré metric,

$$\frac{PU}{PD_1}(0) = r^{-1}. \text{ One has } r^{-1} = \frac{e^\sigma + 1}{e^\sigma - 1},$$

$$\text{since by definition, } \sigma = \int_0^r 2 \frac{ds}{1-s^2} = \log \frac{1+r}{1-r}.$$

This proves the Proposition. □

Let us estimate  $B_{K_R, U_R}$ . We show that

$$B_{K_R, U_R} < (1 - \log c'(H)) e^{1.3R}. \tag{3.21}$$

Together with (3.20) and (3.9), this implies Theorem B1.

The set  $K$  from the Main Lemma is contained in  $K_R$ , thus,  $\log \max_{K_R} |I| \geq \log m$ . Hence, by (3.10),

$$B_{K_R, U_R} < \log M(R) - \log m \tag{3.22}$$

We have shown at the end of 2.7 that

$$\log m \geq \log \Delta_0 - (n^2 - 1) \log M_0 - n^2 \log n - \log 2,$$

$$\Delta_0 = (c'(H))^{6n^3} (c''(H))^{n^2} n^{-62n^3}, \quad M_0 = e^{\frac{2600n^{16}}{c''(H)}} (c'(H))^{-28n^4}.$$

Together with (3.10), (3.22), (3.7) and elementary inequalities this implies (3.21). Theorem B1 is proved modulo inequalities (3.11)-(3.13).  $\square$

**Proof of (3.13) modulo (3.11) and (3.12).** The expression  $V = V_{\alpha, \nu}$  is a linear combination of variations of arguments along the pieces of the path  $\alpha$  that lie either inside  $\beta = \nu$ -neighborhoods of the critical values of  $H$ , or outside  $\overline{D}_3$ . To estimate it from above, we use the following a priori upper bounds of variations.

Let  $a$  be a critical value. By definition, for any curve  $l \subset B$

$$V_l(t - a) = \int_l \frac{|dt|}{|t - a|} = \int_l |dt|_P \frac{(P(B))^{-1}}{|t - a|} \leq |l|_P \max_l \frac{(P(B))^{-1}}{|t - a|}$$

(here by  $|l|_P$  we denote the Poincaré length). The latter ratio is estimated by (3.15):

$$\frac{(P(B))^{-1}}{|t - a|} < |\log |t - a|| + C < 7e^R \log R, \quad \text{whenever } t \in \overline{DP_{R+1}} \quad (3.23)$$

(the last inequality follows from (3.19) and (3.12)). Then by (3.23),

$$V_l(t - a) < 7|l|_P e^R \log R, \quad \text{whenever } l \subset \overline{DP_{R+1}}. \quad (3.24)$$

Analogously, for any critical value  $a$

$$V_l t \leq |l|_P \max_l \frac{(P(B))^{-1}}{|t|} \leq |l|_P \max_l \frac{|t - a|}{|t|} \max_l \frac{(P(B))^{-1}}{|t - a|}.$$

Now let  $l \subset \overline{DP_{R+1}} \setminus \overline{D}_3$ . Then the former maximum in the previous right-hand side is no greater than  $\frac{5}{3}$ , since  $|t| > 3$  on  $l$  and  $|a| \leq 2$ . Substituting this inequality and (3.23) to the same right-hand side yields

$$V_l t < \frac{5}{3} 7|l|_P e^R \log R < 12|l|_P e^R \log R, \quad \text{whenever } l \subset \overline{DP_{R+1}} \setminus \overline{D}_3. \quad (3.25)$$

Let us estimate the expression  $V = V_{\alpha, \nu}$ . By definition, the variations in this expression are taken along the arcs of the path  $\alpha = \lambda^{-1} \alpha_i$  that lie either inside  $D_\nu(a_j)$ , or outside  $\overline{D}_3$  (except for its final arc  $\alpha(\tau', 1] \subset D_\nu(a_i)$ ,  $\alpha(\tau') \in \partial D_\nu(a_i)$ ). By definition, the latter arc coincides with an arc of the path  $\alpha_i$ , and its complement in  $\alpha_i$  is a curve lying in  $\overline{D}_3$  outside the  $\nu$ -neighborhoods of the critical values (see 2.9). Therefore, the previous arcs, where the variations are taken, are disjoint from the path  $\alpha_i$  and thus, are those of the path  $\lambda$ . The first sum in the expression of  $V_{\alpha, \nu}$ , which is  $\nu$  times the sum of the variations along pieces of  $\alpha$  near the critical values, is less than  $7\nu(R + 1)e^R \log R$ . This follows from inequality (3.24) applied to each piece and the inequality  $|\lambda|_P \leq R + 1$ . Analogously, by the latter inequality and (3.25), the second sum in the expression of  $V_{\alpha, \nu}$  is less than  $36(R + 1)e^R \log R$ . The two

previous upper bounds of the sums in  $V_{\alpha,\nu}$  together with (3.7) and inequality  $\nu \leq \frac{1}{16}$  imply that

$$V_{\alpha,\nu} < (36 + 7\nu)(R + 1) \log Re^R < 37R \log Re^R.$$

This proves (3.13).  $\square$

**Proof of (3.11).** Let  $a$  be a critical value of  $H$ ,  $t \in \overline{DP_{R+1}}$ . Let us prove that  $|t| < M_R$ : this will imply (3.11). It follows from definition and (3.15), (3.19) that

$$R + 1 \geq \int_{|t_0-a|}^{|t-a|} \frac{|ds|}{s(|\log s| + C)}, \text{ where } C < \log R. \quad (3.26)$$

By definition,  $|a| \leq 2$ ,  $|t_0| \leq 3$ , so,  $|t_0 - a| \leq 5$ . Suppose  $|t| > 7$  (if not, then the inequality  $|t| < M_R$  follows immediately, since  $M_R > 7$  (by (3.7) and elementary inequalities)). Hence,  $|t - a| > 5$ . Put  $u = \log s$ . Then the latter integral is greater than

$$\int_5^{|t-a|} \frac{ds}{s(\log s + C)} = \log(u + C) \Big|_{\log 5}^{\log |t-a|}$$

By elementary inequalities, the latter right-hand side is greater than

$$\log \log |t - a| - \log(C + 2).$$

This together with (3.26) implies that

$$\log |t - a| < e^{R+1}(C + 2).$$

This together with inequality  $|a| \leq 2$ , (3.19) and elementary inequalities implies (3.11).  $\square$

**Proof of (3.12).** It suffices to show that for any critical value  $a$

$$|t - a| > \beta_R \text{ for any } t \in \overline{DP_{R+1}}. \quad (3.27)$$

It follows from formula for  $\beta_R$  in (3.12), inequality (3.7), choice of  $t_0$  and elementary inequalities that

$$\beta_R < \nu = \frac{c''(H)}{4n^2} \leq |t_0 - a|. \quad (3.28)$$

Thus, if  $|t - a| \geq \nu$ , then inequality (3.27) holds. Let us prove (3.27) assuming that  $|t - a| < \nu$ . To do this, we use the fact that under this assumption the integral in (3.26) is greater than

$$\begin{aligned} \int_{\nu}^{|t-a|} \frac{ds}{s(|\log s| + C)} &= \log(u + C) \Big|_{-\log \nu}^{-\log |t-a|} \\ &> \log \log(|t - a|^{-1}) - \log(-\log \nu + C). \end{aligned}$$

This together with (3.26), (3.19) and elementary inequalities implies (3.27). Inequality (3.12) is proved.  $\square$

### 3.5 Proof of the Geometric Lemma 2.11

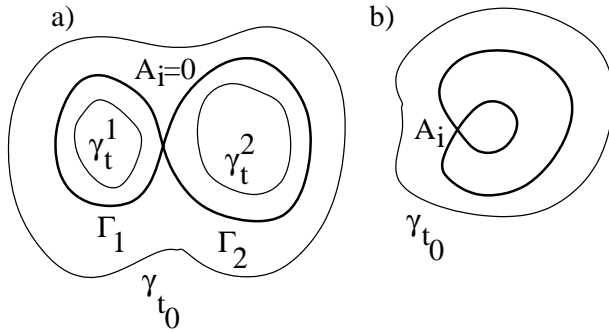
We prove Lemma 2.11 by induction in  $s$ . For  $s = 1$  it is a direct consequence of the definition of vanishing cycle. Indeed, in this case  $[\gamma] = [\gamma_{t_0}]$  is the cycle vanishing along the segment  $[t_0, a_1]$ .

Let the statement of Lemma 2.11 be proved for all  $s < N$ . Let us prove it for  $s = N$ .

Denote by  $A_i$ ,  $i = 1, \dots, N$  the critical points located inside  $\gamma$ . Let  $a_i$  be the corresponding critical values. Without loss of generality suppose that the value  $H(x, y)$  decreases locally when the point  $(x, y)$  in the real plane moves from the oval  $\gamma_{t_0}$  inside the domain bounded by this oval (this may be achieved by changing the sign of  $H$ ). There is a critical point  $A_i$  such that the oval  $\gamma_{t_0}$  extends up to a continuous family of real ovals  $\gamma_t \subset S_t$  on the semiinterval  $(a_i, t_0]$  so that the limit  $\lim_{t \rightarrow a_i} \gamma_t$  is a loop with the base point  $A_N$  (see Fig. 5a, b). This loop is a connected component of a critical level that contains only one singular point of  $H$ , because  $H$  is ultra-Morse. Hence, the limit loop may be either an eight-shaped figure, or a simple loop, see Figures 5a and 5b respectively. Geometric Lemma is proved below in case of the eight-shaped figure, which is a union of two simple loops  $\Gamma_1$  and  $\Gamma_2$  that are disjoint (outside  $A_N$ ) and bound disjoint domains (see Fig. 5a). Another case depicted at Fig. 5b is treated analogously.

Choosing appropriate numeration of the  $a_j$ 's, suppose that  $i = N$ . Without loss of generality we may assume that  $a_N = 0$ ,  $A_N = 0$  (this may be achieved by real translations in the source and target of the map  $H$ ).

When  $t \in \mathbb{R}_+$  passes through 0 to  $\mathbb{R}_-$ , the loop  $\Gamma_1 \Gamma_2$  generates a pair of ovals  $\gamma_t^i$ ,  $i = 1, 2$ , in the real level curve  $H(x, y) = t$ : the oval  $\gamma_t^i$  lies in the domain bounded by the curve  $\Gamma_i$  and tends to  $\Gamma_i$ , as  $t \rightarrow 0$  (see Fig. 5a). Suppose that the curves  $\gamma_{t_0}$ ,  $\Gamma_i$  and  $\gamma_t^i$  are oriented counterclockwise. All the critical points  $A_j$ ,  $j < N$ , are contained in the domains bounded by the ovals  $\gamma_t^i$ . By the induction assumption, each oval  $\gamma_t^i$  satisfies the statement of Lemma 2.11: (2.6) holds for  $\gamma_{t_0}$  replaced by  $\gamma_t^i$ ,  $s = N$  replaced by  $s < N$ .



Limit loops for families of real ovals

Figure 5:

We proceed below the induction step for the case when  $t_0$  is small. In the general case we connect  $t_0$  to a small  $t'_0 \in \mathbb{R}_+$  by a segment  $\alpha' = [t_0, t'_0]$ . The corresponding family of ovals  $\gamma_t$  starting with  $\gamma_{t_0}$  is well-defined over  $\alpha'$ . If  $\alpha'$  does not contain critical values of  $H$ ,

then the decomposition (2.6) of  $\gamma_{t'_0}$  with  $\delta_j$  replaced by the cycles vanishing along  $\alpha' \circ \alpha_j$  (this decomposition for  $\gamma_{t'_0}$  is proved below) extends along  $\alpha'$  to decomposition (2.6) of  $\gamma_{t_0}$ . Otherwise, we replace  $\alpha'$  by its small deformation to the upper half-plane. (Since  $\gamma_t$  is well-defined on the segment, the result of its extension to  $t_0$  along the deformed path  $\alpha'$  is not changed: it is the real oval  $\gamma_{t_0}$ .)

Let us now prove (2.6) for  $t_0$  small. Consider the semicircular path  $\tau(\theta) = t_0 e^{i\theta}$ ,  $\theta \in [0, \pi]$ , which goes around the zero critical value in the upper half-plane. Let  $\delta_N \in H_1(S_{t_0}, \mathbb{Z})$ ,  $\delta'_N \in H_1(S_{-t_0}, \mathbb{Z})$  be the cycles vanishing along the real segments going from  $\pm t_0$  to 0. By definition, the cycle  $\delta'_N$  is obtained as the extension of the cycle  $\delta_N$  along the path  $\tau(\theta)$ . Let us show that the curve  $\gamma_{t_0}$  admits a homotopy by curves  $\gamma_t$  in complex level lines  $H(x, y) = t$  along the path  $t = \tau(\theta)$  so that

$$[\gamma_{-t_0}] = [\gamma_{-t_0}^1] + [\gamma_{-t_0}^2] \pm [\delta'_N]. \quad (3.29)$$

Together with the decompositions (2.6) for  $\gamma_{-t_0}^i$  that is valid by the induction hypothesis, this implies (2.6) for  $\gamma_{t_0}$  and completes the induction step.

To construct the homotopy mentioned above of the oval  $\gamma_{t_0}$ , let us consider the real local analytic coordinate system  $(x', y')$  in a neighborhood of zero critical point such that  $H = x'y'$ ; it exists by the Morse lemma. Then locally near 0, the curves  $\Gamma_i$  are intervals in the new coordinate lines. By the choice of  $t_0$ , we may suppose that the previous neighborhood contains the square  $U$  centered at 0 (in the new coordinates  $(x', y')$ ) whose sides are parallel to the coordinate axes and have length  $2\sqrt{t_0}$ . The curve  $U \cap \gamma_{t_0}$  lies in the first and third quadrants of this chart:  $x', y' > 0$ ;  $x', y' < 0$ . The entire curve  $\gamma_{t_0}$  is split by the points  $b_+ = (\sqrt{t_0}, \sqrt{t_0})$ ,  $b_- = (-\sqrt{t_0}, -\sqrt{t_0})$  into two arcs denoted by  $\Gamma_i(t_0)$ ,  $i = 1, 2$ . Suppose that the intersection of the domain bounded by  $\Gamma_1$  with  $U$  belongs to the quadrant  $x > 0, y < 0$ . Then  $(0, -\sqrt{t_0}), (\sqrt{t_0}, 0) \in \Gamma_1$ , and the curve  $\Gamma_1(t_0)$  is oriented from  $b_-$  to  $b_+$ ;  $(-\sqrt{t_0}, 0), (0, \sqrt{t_0}) \in \Gamma_2$ , and the curve  $\Gamma_2(t_0)$  is oriented from  $b_+$  to  $b_-$  (see Fig.6a). The vanishing cycle  $\delta$  is represented by the circle  $\delta = \{(-\sqrt{t_0}e^{i\psi}, \sqrt{t_0}e^{-i\psi}) \mid \psi \in [0, 2\pi]\}$ .

Our goal is to construct the family  $\gamma_{\tau(\theta)}$  and then check (3.29). Below we construct the homotopy  $\Gamma_i(\tau(\theta))$  of each arc  $\Gamma_i(t_0)$  along the path  $\tau(\theta)$  (as a family of arcs in complex level curves  $H = \tau(\theta)$ ) so that

- 1)  $\Gamma_i(-t_0) = \gamma_{-t_0}^i$ ;
- 2) the arc  $\Gamma_1(\tau(\theta))$  starts at  $(-\sqrt{t_0}e^{i\theta}, -\sqrt{t_0})$  and ends at  $(\sqrt{t_0}, \sqrt{t_0}e^{i\theta})$ ;
- 3) the arc  $\Gamma_2(\tau(\theta))$  starts at  $(\sqrt{t_0}e^{i\theta}, \sqrt{t_0})$  and ends at  $(-\sqrt{t_0}, -\sqrt{t_0}e^{i\theta})$ .

Then we put

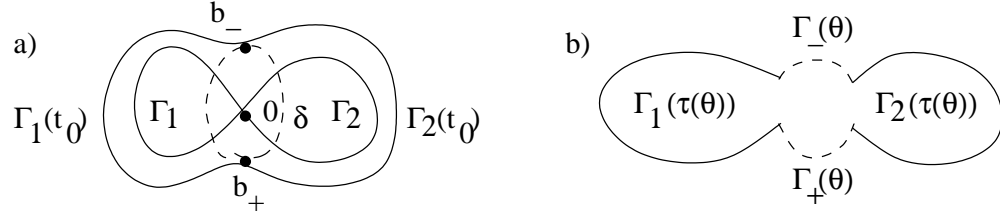
$$\Gamma_{\pm}(\theta) = \{(\pm\sqrt{t_0}e^{i\phi}, \pm\sqrt{t_0}e^{i(\theta-\phi)}) \mid 0 \leq \phi \leq \theta\} \text{ (with a natural orientation),}$$

and a representative of the class  $[\gamma_{\tau(\theta)}]$  may be constructed as a product of four curves:

$$\gamma_{\tau(\theta)} = \Gamma_1(\tau(\theta))\Gamma_+(\theta)\Gamma_2(\tau(\theta))\Gamma_-(\theta).$$

By construction, the curves  $\gamma_{\tau(\theta)} \subset \{H(x, y) = \tau(\theta)\}$  are closed (with a well-defined orientation) and depend continuously on  $\theta$ ,  $\gamma_{\tau(0)} = \gamma_{t_0}$ . Note that  $[\delta'_N] = [\Gamma_+(\pi)\Gamma_-(\pi)]$ . This implies (3.29) and proves Lemma 2.11 modulo existence of families  $\Gamma_i(\tau(\theta))$  satisfying assumptions 1)-3) above.

We construct the family  $\Gamma_1(\tau(\theta))$  only, the family  $\Gamma_2(\tau(\theta))$  is constructed analogously. To do this, consider an arbitrary increasing parametrization  $\psi : u \in [0, 1] \mapsto \Gamma_1$  of the oriented



Homotopy  $\{\gamma_{\tau(\theta)}\}$ : a)  $\theta=0$ ,  $\gamma_{\tau(0)}$  is a real curve;

b) intermediate  $\theta$ ;  $\gamma_{\tau(\theta)}$  is a curve on a Riemann surface in  $\mathbb{C}^2$

Figure 6:

curve  $\Gamma_1$ ,  $\psi' \neq 0$ ,  $\psi(0) = \psi(1) = 0$ . Recall that the local starting branch (going from 0) of  $\Gamma_1$  lies in the negative  $y'$ - semiaxis, so, the coordinate  $-y'$  increases locally along this branch. Analogously, the coordinate  $-x'$  increases locally along the final branch entering 0 of  $\Gamma_1$ . Let us choose the previous parametrization so that

$$u = -y' \circ \psi(u) \text{ near } u = 0; \quad u = -x' \circ \psi(u) + 1 \text{ near } u = 1.$$

The mapping  $\psi$  extends up to a locally invertible  $C^\infty$  mapping of  $\mathbb{C}^2$ - complex neighborhood of the initial parameter segment  $[0, 1]$  (we consider that the coordinates in the new parameter space  $\mathbb{C}^2$  are  $(u, v)$ , and the previous segment  $[0, 1]$  lies in the complex  $u$ - axis). One can choose the previous extension of  $\psi$  so that

a) the previous equalities hold in complex neighborhoods of the points  $(0, 0)$ ,  $(1, 0)$ :

$$u = -y' \circ \psi(u, v) \text{ near } (0, 0); \quad u = -x' \circ \psi(u, v) + 1 \text{ near } (1, 0); \quad (3.30)$$

b) the level curves of the pull-back  $H \circ \psi$  of  $H$  (except for the lines  $\{u = 0, 1\} \subset \{H \circ \psi = 0\}$ ) are transversal to the lines  $u = \text{const}$ .

Then  $(u, H)$  are well defined global coordinates on the complement of the parameter domain to the latter pair of lines  $u = 0, 1$ . Let  $\widetilde{\Gamma_1(t_0)}$  be the lifting to the parameter domain of the arc  $\Gamma_1(t_0)$ ,  $\widetilde{\Gamma_1(\tau(\theta))}$  be its image under the mapping preserving the coordinate  $u$  and multiplying the coordinate  $H \circ \psi$  by  $e^{i\theta}$ . The arc  $\Gamma_1(\tau(\theta)) = \psi(\widetilde{\Gamma_1(\tau(\theta))})$  is the one we are looking for. Indeed, it lies in the complex level curve  $H = \tau(\theta)$  by construction. It starts at  $(-\sqrt{t_0}e^{i\theta}, -\sqrt{t_0})$  and ends at  $(\sqrt{t_0}, \sqrt{t_0}e^{i\theta})$  by (3.30) and the equality  $H = x'y'$ . It follows from construction and (3.30) that  $\Gamma_1(-t_0) = \gamma_{-t_0}^1$ . Lemma 2.11 is proved.

## 4 Estimates of the number of zeros of Abelian integrals near the critical values

In this section we prove Theorem A2, see 1.8, and Theorem B2, stated below. Together with Theorem A1 (whose proof is completed in section 2) Theorem A2 implies Theorem A.

Together with Theorem B1 ( whose proof is completed in section 3), Theorem B2 implies Theorem B.

We have three statements to discuss:

1. Theorem A2 in the case when the endpoints of the interval considered are all finite;
2. Theorem A2 in the case when one of these endpoints is infinite;
3. Theorem B2.

These statements will be referred to as cases 1,2,3 below.

It appears that cases 1 and 3 are very close to each other.

#### 4.1 Argument principle, KRY theorem and Petrov's method

All the three cases are treated in a similar way. We want to apply the argument principle.

The estimates near infinity are based on the argument principle only. The estimates near finite critical points use the Petrov's method that may be considered as a generalization of the argument principle for multivalued functions. The increment of the argument is estimated through the Bernstein index of the integral, bounded from above in the previous sections. The relation between these two quantities is the subject of the Khovanskii-Roitman-Yakovenko (KRY) theorem and Theorems 4.3, 4.4 stated below. It seems surprising that these theorems were not discovered in the classical period of the development of complex analysis. The preliminary version of the theorem was proved in [19], the final one in [11]. One of the two inequalities in this theorem is proved by the second author (Yu.S.Ilyashenko, [8]) in a stronger form (Theorem 4.4 below): an explicit formula for the constant in the estimate is written, and  $\mathbb{C}$  is replaced by an arbitrary Riemann surface.

At this spot we begin the proof of Theorem A2 in case 1. Recall the statement of the theorem in case 1.

**Theorem A2 (Case 1).** *Let  $a \neq \infty, b \neq \infty, .$  Then*

$$\#\{t \in (a, l(t_0)) \cup (r(t_0), b) \mid I(t) = 0\} < (1 - \log c') e^{\frac{4800}{c''} n^4},$$

where  $l(t_0)$  and  $r(t_0)$  are the same as at the beginning of 1.7.

We will prove that

$$\#\{t \in (a, l(t_0)) \mid I(t) = 0\} < \frac{1}{2} (1 - \log c') e^{\frac{4800}{c''} n^4}. \quad (4.1)$$

Similar estimate for  $(r(t_0), b)$  is proved in the same way. These two estimates imply Theorem A2.

Let  $\Pi = \Pi(a)$  be the same as in (1.4).

**4.1 Lemma** *Inequality (4.1) holds provided that in (4.1) the interval  $(a, l(t_0))$  is replaced by  $\Pi$ .*

Lemma 4.1 implies (4.1) because  $(a, l(t_0)) \subset \Pi$ . Let

$$\Pi_\psi = \{t \in \Pi \mid \psi \leq |t - a| \leq \nu\}$$

**4.2 Lemma** *Lemma 4.1 holds provided that in (4.1) the domain  $\Pi$  is replaced by  $\Pi_\psi$ .*



Lemma 4.2 implies Lemma 4.1, because

$$\Pi = \cup_{\psi>0} \Pi_\psi.$$

**Proof of Lemma 4.2.** The proof of this lemma occupies this and the next four subsections. We have

$$\partial\Pi_\psi = \Gamma_1\Gamma_2\Gamma_3\Gamma_4.$$

As sets, the curves  $\Gamma_j$  are defined by the formulas below; the orientation is defined separately:

$$\Gamma_1 = \{t \mid |t - a| = \nu, |\arg(t - a)| \leq 2\pi\} = \Gamma_a$$

$$\Gamma_3 = \{t \mid |t - a| = \psi, |\arg(t - a)| \leq 2\pi\}$$

$$\Gamma_{2,4} = \{t \mid \psi \leq |t - a| \leq \nu, \arg(t - a) = \pm 2\pi\}.$$

The curve  $\Gamma_1$  is oriented counterclockwise,  $\Gamma_2$  is oriented from the right to the left,  $\Gamma_3$  is oriented clockwise,  $\Gamma_4$  is oriented from the left to the right.

Let  $\#\{t \in (a + \psi, l(t_0)) \mid I(t) = 0\} = Z_\psi$ . Denote by  $R_\Gamma(f)$  the increment of the argument of a holomorphic function  $f$  along a curve  $\Gamma$  ( $R$  of Rouchet). Recall that  $V_\Gamma(f)$  denotes the variation of the argument of  $f$  along  $\Gamma$ . Obviously,  $|R_\Gamma(f)| \leq V_\Gamma(f)$ .  $\square$

In assumption that  $I \neq 0$  on  $\partial\Pi_\psi$ , the argument principle implies that

$$Z_\psi \leq \frac{1}{2\pi} R_{\partial\Pi_\psi}(I) \leq \frac{1}{2\pi} \sum_1^4 R_{\Gamma_j}(I). \quad (4.2)$$

The first term in this sum is estimated by the modified KRY theorem, the second and the fourth one by the Petrov method, the third one by the Mardesic theorem. The case when the above assumption fails is treated in 4.3.

## 4.2 Bernstein index and variation of argument

The first step in establishing a relation between variation of argument and the Bernstein index was done by the following KRY theorem.

Let  $U$  be a connected and simply connected domain in  $\mathbb{C}$ ,  $\Gamma \subset U$  be a (nonoriented) curve,  $f$  be a bounded holomorphic function on  $U$ .

**KRY theorem, [11]** *For any tuple  $U, \Gamma \subset U$  as above and a compact set  $K \subset U$  there exists a geometric constant  $\alpha = \alpha(U, K, \Gamma)$ , such that*

$$V_\Gamma(f) \leq \alpha B_{K,U}(f).$$

In [11] an upper estimate of the Bernstein index through the variation of the argument along  $\Gamma = \partial U$  is given; we do not use this estimate. On the contrary, we need an improved version of the previous theorem with  $\alpha$  explicitly written and  $U$  being a domain on a Riemann surface. These two goals are achieved in the following two theorems.

Let  $|\Gamma|$  be the length, and  $\kappa(\Gamma)$  be the total curvature of a curve on a surface endowed with a Riemann metric.

**4.3 Theorem** Let  $\Gamma \subset U'' \subset U' \subset U \subset \mathbb{C}$  be respectively a curve, and three open sets in  $\mathbb{C}$ . Let  $f : U \rightarrow \mathbb{C}$  be a bounded holomorphic function,  $f|_{\Gamma} \neq 0$ . Let  $\varepsilon < \frac{1}{2}$  and the following gap conditions hold:

$$\rho(\Gamma, \partial U'') \geq \varepsilon, \quad \rho(U'', \partial U') \geq \varepsilon, \quad \rho(U', \partial U) \geq \varepsilon. \quad (4.3)$$

Let  $D > 1$  and the following diameter conditions hold:

$$\text{diam}_{\text{int}} U'' \leq D, \quad \text{diam}_{\text{int}} U' \leq D. \quad (4.4)$$

Then

$$V_{\Gamma}(f) \leq B_{U'', U}(f) \left( \frac{|\Gamma|}{\varepsilon} + \kappa(\Gamma) + 1 \right) e^{\frac{5D}{\varepsilon}}. \quad (4.5)$$

**4.4 Theorem** Let  $\Gamma \subset U'' \subset U' \subset U \subset W$  be respectively a curve, and three open sets in a Riemann surface  $W$ . Let  $f : U \rightarrow \mathbb{C}$  be a bounded holomorphic function,  $f|_{\Gamma} \neq 0$ . Let  $\pi : W \rightarrow \mathbb{C}$  be a projection which is locally biholomorphic, and the metric on  $W$  is a pullback of the Euclidean metric in  $\mathbb{C}$ . Let  $\varepsilon < \frac{1}{2}$  and the following gap conditions hold:

$$\pi\text{-gap}(\Gamma, U'') \geq \varepsilon, \quad \pi\text{-gap}(U'', U') \geq \varepsilon, \quad \pi\text{-gap}(U', U) \geq \varepsilon. \quad (4.6)$$

Let  $D > 1$  and the following diameter conditions hold:

$$D > 1, \quad \text{diam}_{\text{int}} U'' \leq D, \quad \text{diam}_{\text{int}} U' \leq D \quad (4.7)$$

Then inequality (4.5) holds.

These theorems are proved in [8].

Recall that intrinsic diameter and  $\pi$ -gap are defined in 1.6.

We can now estimate from above the first term in the sum (4.2). The estimate works in both cases when  $a$  is finite or infinite. Let for simplicity, as in (1.5),

$$A = e^{\frac{n^4}{c''}}.$$

**4.5 Lemma** Let  $H$  be a balanced polynomial of degree  $n + 1 \geq 3$ . Let  $I$  be the same integral as in (1.1). Let  $K$  be a compact set mentioned in the Main Lemma, and  $\Gamma_1 = \Gamma_a$  be the same as in this Lemma ( $a$  may be infinite). Then

$$V_{\Gamma_1}(I) < (1 - \log c') A^{4700}. \quad (4.8)$$

**Proof** The lemma follows immediately from Theorem 4.4 and the Main Lemma. To apply Theorem 4.4, let us take  $I$  for  $f$ , the universal cover over  $B$  for  $W$  with the natural projection  $\pi : W \rightarrow \mathbb{C}$  and metric induced from  $\mathbb{C}$  by this projection. This metric on  $W$  is called Euclidean. Let  $K$  and  $U$  be the same as in the Main Lemma. Take this  $U$  for the domain  $U$  to apply Theorem 4.4. Recall that  $U$  is the minimal simply connected domain that contains the  $\frac{\nu}{2}$ -neighborhood of  $K$  in  $U$  in the Euclidean metric on  $W$ ,  $\nu$  is the same as in (1.2). Let  $\varepsilon = \frac{\nu}{6}$ , that is

$$\varepsilon = \frac{c''}{24n^2}. \quad (4.9)$$

Let  $U''$  and  $U'$  be the minimal simply connected domains in  $W$  that contain  $\varepsilon$ -neighborhood of  $K$  and  $U''$  respectively. Note that  $\Gamma_1 = \Gamma \subset K$ . Then gap condition (4.6) with  $\varepsilon$  from (4.9)

holds. Moreover,  $\text{diam}_{\text{int}}U \leq \text{diam}_{\text{int}}U' + 2\varepsilon \leq \text{diam}_{\text{int}}K + 4\varepsilon$ . Hence, diameter condition (4.7) holds with  $D < 38n^2$  by (1.11). Thus

$$e^{\frac{5D}{\varepsilon}} < A^c, \text{ where } A = e^{\frac{n^4}{c''}}, c = 5 \times 38 \times 24 < 4600.$$

This factor  $A^c$  is the largest one in the estimate for  $V_{\Gamma_1}(I)$ .

By inequality (1.13) from the Main Lemma,  $B_{K,U} \leq (1 - \log c')A^2$ . By the monotonicity of Bernstein index (that follows directly from the definition),  $B_{U'',U} < B_{K,U}$ . At last,

$$\frac{|\Gamma_1|}{\varepsilon} + \kappa(\Gamma_1) + 1 = 24\pi + 4\pi + 1 \ll A.$$

Now, inequality (4.5) proves the lemma.  $\square$

The Corollary below is used in the next subsection.

**4.6 Remark** Lemma 4.5 remains valid if in its hypothesis the integral  $I$  is replaced by an integral  $J$  over the cycle vanishing at the critical value  $a$  of  $H$ . The proof of this modified version of Lemma 6.3 repeats that of the original one with the following change: we use the Modified Main Lemma instead of the Main Lemma.

**4.7 Corollary** *Suppose that the integral  $J$  with a real integrand  $\omega$  is taken over a local vanishing cycle  $\delta_t$  corresponding to the real critical value  $a$ . Then the number of zeros of  $J$  in the disk centered at  $a$  of radius  $\nu = \frac{c''}{4n^2}$  admits the following upper estimate:*

$$N_J := \#\{t \in \mathbb{C} \mid |t - a| < \nu, J(t) = 0\} \leq \frac{1}{2\pi}(1 - \log c')A^{4700} \quad (4.10)$$

This follows from the modified Lemma 4.5 and the argument principle.

### 4.3 Application of the Petrov's method

The Petrov's method applied below is based on a remark that the magnitude of the increment of the argument of a nonzero function along an oriented curve is no greater than the number of zeros of the imaginary part of this function increased by 1 and multiplied by  $\pi$ . Indeed, at any half circuit around zero, a planar curve crosses an imaginary axis at least once. The method works when the imaginary part of a function appears to be more simple than the function itself.

Let  $\delta_t \in H_1(t)$  be the local vanishing cycle at the point  $a$ . Let  $\omega$  be the same real form as in integral (1.1). Let  $J$  be the germ of integral  $J(t) = \int_{\delta_t} \omega$  along the cycle  $\delta_t$ , which is a local vanishing cycle at  $t = a$ . Note that  $J$  is single-valued in any simply connected neighborhood of  $a$  that contains no other critical values of  $H$ . Let  $l_0 = (\gamma_t, \delta_t)$  be the intersection index of the cycles  $\gamma_t$  and  $\delta_t$ . As the cycle  $\gamma_t$  is real and  $H$  is ultra-Morse,  $l_0$  may take values  $\pm 1, \pm 2$  only (Lemma 2.21). Let

$$\Gamma_0 = \{t \in \mathbb{R} \mid te^{2\pi i} \in \Gamma_2\}.$$

Then by the Picard-Lefschetz theorem

$$I|_{\Gamma_2} = (I + l_0 J)|_{\Gamma_0}, \quad I|_{\Gamma_4} = (I - l_0 J)|_{\Gamma_0}.$$

**4.8 Proposition** *The integral  $J$  is purely imaginary on the real interval  $(a, b)$ .*

**Proof** Recall that the form  $\omega$  and the polynomial  $H$  are real. Then

$$J(t) = -\overline{J(\bar{t})}.$$

Indeed,  $\omega = Q(x, y)dx$ . The involution  $\mathbf{i} : (x, y) \mapsto (\bar{x}, \bar{y})$  brings the integral  $J(t) = \int_{\delta_t} Qdx$  to  $\int_{\mathbf{i}\delta_t} \overline{Qd\bar{x}} = \int_{-\delta_{\bar{t}}} \overline{Qd\bar{x}} = -\int_{\delta_{\bar{t}}} \overline{Qdx} = -\overline{J(\bar{t})}$ . On the other hand, for real  $t$  we have  $t = \bar{t}$  and  $\delta_{\bar{t}} = \delta_t$ . Hence,  $J(t) = -\overline{J(t)}$  for  $t \in (a, b)$ . This implies Proposition 6.1.  $\square$

**4.9 Corollary**

$$\text{Im} I |_{\Gamma_{2,4}} = \pm l_0 J |_{\sigma}.$$

**Proof** This follows from Proposition 4.8, Picard-Lefschetz theorem and the reality of  $I$  on  $\sigma$ .  $\square$

Suppose now that  $I$  has no zeros on  $\Gamma_2$  and  $\Gamma_4$ . Then

$$|R_{\Gamma_{2,4}}(I)| \leq \pi(1 + N), \text{ where } N = \#\{t \in \Gamma_0 \mid J(t) = 0\}. \quad (4.11)$$

Obviously,  $N \leq N_J$ , see (4.10). The right hand side of this inequality is already estimated from above in Corollary 4.7. Hence,

$$|R_{\Gamma_{2,4}}(I)| \leq \pi(1 - \log c')A^{4700}.$$

Suppose now that  $I$  has zeros on  $\Gamma_2$  (hence on  $\Gamma_4$ , by Proposition 4.8). Indeed, its real part is the same at the corresponding points of  $\Gamma_2, \Gamma_0, \Gamma_4$ , and the imaginary parts of  $I|_{\Gamma_2}$  and  $I|_{\Gamma_4}$  are opposite at the corresponding points.) In this case we replace the domain  $\Pi_\psi$  by  $\Pi'_\psi$  defined as follows.

The curves  $\Gamma_{2,4}$  should be modified. A small segment of  $\Gamma_2$  centered at zero point of  $I$  that contains no other zeros of  $J$ , should be replaced by an upper half-circle having this segment as a diameter and containing no zeros of  $J$ . A similar modification should be done for  $\Gamma_4$  making use of lower half-circles. Denote the modified curves by  $\Gamma'_{2,4}$ . Let  $\Pi'_\psi$  be the domain bonded by the curve

$$\partial\Pi'_\psi = \Gamma_1\Gamma'_2\Gamma_3\Gamma'_4. \quad (4.12)$$

It contains  $\Pi_\psi$ , and we will estimate from above the number of zeros of  $I$  in  $\Pi'_\psi$  still using the argument principle. The increment of  $\arg I$  along  $\Gamma_1$  is already estimated in 4.2. Here we give an upper bound for the increment of  $\arg I$  along  $\Gamma'_{2,4}$ . The increment along  $\Gamma_3$  is estimated in the next subsection.

**4.10 Proposition** *Let  $N$  be the same as in (4.11). Then*

$$|R_{\Gamma'_2}(I)| \leq \pi(2N + 1). \quad (4.13)$$

**Proof** Let  $I$  have zeros  $b_j \in \Gamma_2$ ,  $j = 1, \dots, k$ , the number of occurrence of  $b_j$  in this list equals its multiplicity. Note that

$$\text{Im } I|_{\Gamma_2} = \pm l_0 J \circ \pi, \quad l_0 = \langle \delta, \gamma \rangle \neq 0. \quad (4.14)$$

Hence, at the points  $b_j$ ,  $J$  has zeros of no less multiplicity than  $I$ . Hence, the total multiplicity  $k'$  of zeros of  $J$  at the points  $b_j \in \Gamma_2$ ,  $j = 1, \dots, k$ , is no less than  $k$ . Let  $J$  have  $s$  zeros on  $\Gamma'_2$ . We have:  $k' \geq k$ ,  $s \leq N - k' \leq N - k$ . Let  $\sigma_1, \dots, \sigma_q$ ,  $q \leq k + 1$ , be the open intervals into which the curve  $\Gamma_2$  is divided by the points  $b_j$ . Let  $s_j$  be the number of zeros of  $J$  on  $\sigma_j$ ,  $\sum_1^q s_j = s$ . Let

$$R_j = R_{\sigma_j}(I).$$

Then

$$R_j \leq \pi(s_j + 1).$$

Hence,

$$|R_{\Gamma'_2}(I)| \leq \pi(k + \sum_1^q (s_j + 1)) \leq \pi(2k + 1 + s) \leq \pi(2k' + 1 + s) \leq \pi(2N + 1). \quad (4.15)$$

□

## 4.4 Application of the Mardesic theorem

**4.11 Proposition** *Let  $I$  be the integral (1.1), and  $\Gamma_3$  be the same as in 4.1. Then for  $\psi$  small enough,*

$$|R_{\Gamma_3}(I)| \leq \pi(4n^4 + 1). \quad (4.16)$$

**Proof** Let  $J$  and  $l_0$  be the same as in the previous subsection. Let  $a = 0$ , and  $I(e^{2\pi i t})$  means the result of the analytic extension of  $I$  from a value  $I(t)$  along a curve  $e^{2\pi i \varphi t}$ ,  $\varphi \in [0, 1]$ . By the Picard-Lefschetz theorem, for small  $t$

$$I(e^{2\pi i t}) = I(t) + l_0 J(t).$$

Consider the function

$$Y(t) = I(t) - l_0 \frac{\log t}{2\pi i} J(t).$$

This function is single-valued because the increments of both terms  $I$  and  $Y$  under the analytic extension over a circle centered at 0 cancel. The function  $I$  is bounded along any segment ending at zero, and  $J$  is holomorphic at zero, with  $J(0) = 0$ . Hence,  $Y$  is holomorphic and grows no faster than  $\log |t|$  in a punctured neighborhood of zero. (In fact, it is bounded in the latter neighborhood:  $|J(t) \log t| \leq c|t| |\log t| \rightarrow 0$ , as  $t \rightarrow 0$ .) By the removable singularity theorem, it is holomorphic at zero. Hence,

$$I(t) = Y(t) + l_0 \frac{\log t}{2\pi i} J(t) \quad (4.17)$$

with  $Y$  and  $J$  holomorphic. We claim that *the increment of the argument of  $I$  along  $\Gamma_3$  for  $\psi$  small is bounded from above through  $\text{ord}_0 J$ , the order of zero of  $J$  at zero.* The latter order is estimated from above by the following theorem by Mardesic:

**4.12 Theorem** [12]. *The multiplicity of any zero of the integral  $I$  (or  $J$ ) taken at a point where the integral is holomorphic does not exceed  $n^4$ .*

The function (4.17) is multivalued. The proof of the latter claim is based on the following simple remark. Let  $f_1, f_2$  be two continuous functions on a segment  $\sigma \subset \mathbb{R}$ , and  $|f_1| \geq 2|f_2|$ . Then  $R_\sigma(f_1 + f_2) \leq R_\sigma(f_1) + \frac{2\pi}{3}$ . Indeed, the value  $R_\sigma(f_1 + \varepsilon f_2)$  cannot change more than by  $\frac{2\pi}{3}$ , as  $\varepsilon$  ranges over the segment  $[0, 1]$ .

To complete the proof of Proposition 4.11, we need to consider three cases. Let  $\nu = \text{ord}_0 Y$ ,  $\mu = \text{ord}_0 J$ ,  $f(\varphi) = Y(\psi e^{2\pi i \varphi})$ ,  $g(\varphi) = \left(J \frac{\log}{2\pi i}\right)(\psi e^{2\pi i \varphi})$ . Note that  $\mu \leq n^4$ .

Case 1:  $\nu < \mu$ . Then, for  $\psi$  small,  $2|g| \leq |f|$ . By the previous remark, applied to  $f_1 = f$ ,  $f_2 = g$ , we get

$$|R_{\Gamma_3}(I)| \leq \pi(4\nu + 1) \leq \pi(4n^4 + 1).$$

Case 2:  $\nu = \mu$ . Then, for  $\psi$  small,  $2|f| \leq |g|$ , because of the logarithmic factor in  $g$ . In the same way as before, we get

$$|R_{\Gamma_3}(I)| \leq \pi(4\mu + 1) \leq \pi(4n^4 + 1).$$

Case 3:  $\nu > \mu$ . In the same way, as in Case 2, we get (4.16).  $\square$

## 4.5 Proof of Theorem A2 in case 1 (endpoints of the interval considered are finite)

**Proof** It is sufficient to prove Lemma 4.2. We prove a stronger statement

$$N(I, \Pi'_\psi) := \#\{t \in \Pi'_\psi \mid I(t) = 0\} < \frac{1}{2}(1 - \log c')A^{4700} \quad (4.18)$$

By the argument principle

$$2\pi N(I, \Pi'_\psi) \leq V(\Gamma_1) + |R_{\Gamma_2}(I)| + |R_{\Gamma_3}(I)| + |R_{\Gamma_4}(I)| \quad (4.19)$$

The first term in the r.h.s is estimated in (4.8). The second and the fourth terms are estimated from above in (4.15). The third term is estimated in (4.16). Altogether this proves (4.18), hence, Lemma 4.2 and implies a stronger version of (4.1):

$$\#\{t \in \Pi'_\psi \mid I(t) = 0\} < \frac{1}{2}(1 - \log c')A^{4700}.$$

This proves Theorem A2 in case 1.  $\square$

## 4.6 Proof of Theorem B2

**Theorem B2.** *For any real ultra-Morse polynomial  $H$ , any family  $\Gamma$  of real ovals of  $H$ , and any  $l$ , let  $\Pi(a)$  and  $\Pi(b)$ ,  $D(l, a)$  and  $D(l, b)$  be the same domains, as in (1.4). Let  $I$  be the analytic extension to  $W$  of the integral (1.1) over the ovals of the family  $\Gamma$ :*

$$\int_{\gamma_t} \omega = I(t), \quad \gamma_t \in \Gamma.$$

Then the number of zeros of  $I$  in  $D(l, a)$  and  $D(l, b)$  (denoted by  $N(l, H)$ ), is no greater than

$$N(l, H) \leq (1 - \log c'(H))e^{4700\frac{n^4}{c''} + \frac{481l}{c''}}.$$

**Proof** We will prove the theorem for the case when  $a = a(t_0)$  is a logarithmic branch point of the integral  $I$  at the left end of the segment  $\sigma(t_0)$ . The case of the right end is treated in the same way. The case when  $a(t_0)$  is a critical value of  $H$  which is not a singular point of the integral  $I$ , is even more elementary. In this case the integral is univalent in a small neighborhood of  $a$ , the number of zeros to be estimated does not depend on  $l \geq 1$ , and the estimate follows from TheoremA2.

Let for simplicity  $D(l) = D(l, a)$ . For any  $\psi \in (0, \nu)$  consider the set  $\Pi'_\psi \subset W$ , see (4.12). Let

$$\Pi'_{\psi, l} = \{re^{i\varphi} + a(\Gamma) \in W \mid re^{\frac{i\varphi}{l}} + a(\Gamma) \in \Pi'_\psi\}.$$

Let  $\Gamma_{1, l}, \Gamma'_{2, l}, \Gamma_{3, l}, \Gamma'_{4, l}$  be the curves defined by the relations:

$$\partial\Pi'_{\psi, l} = \Gamma_{1, l}\Gamma'_{2, l}\Gamma_{3, l}\Gamma'_{4, l};$$

$$\pi\Gamma_{j, l} = \pi\Gamma_j, \quad j = 1; 3;$$

$$\pi\Gamma'_{j, l} = \pi\Gamma'_j, \quad j = 2; 4.$$

Let  $R_\Gamma(f)$  and  $V_\Gamma(f)$  be the same as in 4.1. Then, by the argument principle

$$2\pi N(l, H) \leq V_{\Gamma_{1, l}}(I) + |R_{\Gamma'_{2, l}}(I)| + |R_{\Gamma_{3, l}}(I)| + |R_{\Gamma'_{4, l}}(I)|. \quad (4.20)$$

The four terms in the right hand side are estimated in a similar way as the corresponding terms in (4.19). The last three terms are in fact already estimated:

$$|R_{\Gamma'_{j, l}}(I)| \leq \pi(2N + 1), \quad j = 2, 4, \quad (4.21)$$

where  $N$  is the same as in (4.11);

$$|R_{\Gamma_{3, l}}(I)| \leq \pi(4n^4l + 1). \quad (4.22)$$

□

**4.13 Proposition** *Inequality (4.21) holds.*

**Proof** The proposition is proved in the very same way as Proposition 4.10 with the only difference: (4.14) should be replaced by

$$\text{Im } I|_{\Gamma_{j, l}} = \pm ll_0 J \circ \pi|_{\Gamma_0}.$$

The factor  $l$  in the r.h.s. does not change the number of zeros. □

Inequality (4.22) is proved in the same way as (4.16) with the only difference that the increment of the argument of  $t$  along  $\Gamma_{3, l}$  is now  $4\pi l$ .

**4.14 Proposition** Let  $A = e^{\frac{n^4}{c''}}$ . Then

$$V_{\Gamma_1, l}(I) \leq (1 - \log c'(H))A^{4700} e^{\frac{481l}{c''}} \quad (4.23)$$

**4.15 Remark** Inequalities (4.21), (4.22), (4.23) together prove Theorem B2.

**Proof of proposition 4.14.** The proof follows the same lines as that of Lemma 4.5. We will estimate the variation of argument under consideration making use of Theorem 4.4. For this we need first to choose the curve  $\Gamma$  and domains  $U'', U', U$ . Let

$$\Gamma = \Gamma_{1, l} = \{a + \nu e^{i\varphi} \mid \varphi \in [-2\pi l, 2\pi l]\}.$$

Take the same  $\varepsilon$  as in (4.9). For any set  $A \subset W$  take  $A^\varepsilon$  to be the  $\varepsilon$ -neighborhood of  $A$  in the Euclidean metric of  $W$ , and  $\overline{A^\varepsilon}$  be the minimal simply connected domain that contains  $A^\varepsilon$ . Let  $K$  be the same as in the Main Lemma. Take

$$U'' = \overline{(K \cup \Gamma)^\varepsilon}, \quad U' = \overline{(K \cup \Gamma)^{2\varepsilon}}, \quad U = \overline{(K \cup \Gamma)^{3\varepsilon}}.$$

Note that for any point  $p \in K \cup \Gamma$ , the  $6\varepsilon$ -neighborhood of  $p$  in  $W$  is bijectively projected to a  $6\varepsilon$ -disk in  $\mathbb{C}$ . Hence, the gap condition (4.6) holds for  $\Gamma, U'', U', U$  so chosen.

Note that  $K \cap \Gamma = \Gamma_1 \neq \emptyset$ . Hence, the set  $K \cup \Gamma$ , as well as  $U'', U', U$  is path connected. Then we have:

$$\text{diam } K \leq 36n^2$$

by (1.11),

$$\text{diam } \text{int}(K \cup \Gamma) \leq 36n^2 + 4\pi l\nu := D_1,$$

$$\text{diam } \text{int}U'' \leq D_1 + 2\varepsilon,$$

$$\text{diam } \text{int}U' \leq D_1 + 4\varepsilon.$$

Hence, diameter condition (4.7) holds with

$$D_2 = 36n^2 + 16l\nu = 36n^2 + \frac{4lc''}{n^2},$$

or with  $D = 36n^2 + \frac{4l}{n^2} \geq D_2$  because  $c'' \leq 1$ .

Let us now estimate from above the Bernstein index  $B_1 = B_{U'', U}(I)$ . Let  $U_0$  be the domain denoted by  $U$  in the Main Lemma.

Then  $K \subset U'', U_0 \subset U$ . Let  $B_0 = B_{K, U_0}(I)$  be the Bernstein index estimated in the Main Lemma. By (1.13),

$$B_0 < (1 - \log c')A^2, \quad A = e^{\frac{n^4}{c''}}.$$

□

**4.16 Proposition**

$$B_1 \leq B_0 + \log(4l + 1).$$



**Proof** By definition,

$$B_1 = \log \frac{M_1}{m_1}, \quad B_0 = \log \frac{M_0}{m},$$

where  $M_1 = \max_{\overline{U}} |I|$ ,  $m_1 = \max_{\overline{U''}} |I|$ ,  $M_0 = \max_{\overline{U_0}} |I|$ ,  $m = \max_K |I|$ . Note that  $K \subset U''$ , hence,  $m \leq m_1$ .

On the other hand, let

$$M_J = \max |J| \text{ on the closure of } D_{\nu+3\varepsilon}(a).$$

By definition,  $\Gamma^{3\varepsilon} \subset U_0$ ,  $\pi\Gamma^{3\varepsilon} \subset D_{\nu+3\varepsilon}(a)$ ,  $U$  is the minimal simply connected domain containing  $U_0 \cup \Gamma^{3\varepsilon}$ . By the Picard-Lefschetz theorem and Lemma 2.21

$$M_1 \leq M_0 + |l_0|lM_J, \quad |l_0| \leq 2.$$

Let us estimate the integral  $J$  from above. Over each point of  $\partial D_{\nu+3\varepsilon}(a)$  there are two points of  $\partial(\Gamma_1^{3\varepsilon}) \subset \overline{U_0}$ , where  $\Gamma_1 = \Gamma_a$  is the same, as in (4.1). The difference of the values of  $I$  at the two latter points is equal to  $\pm l_0 J$ ,  $0 < |l_0| \leq 2$ . Therefore,

$$M_J \leq 2M_0. \text{ Hence,}$$

$$M_1 \leq M_0(4l + 1),$$

$$B_1 = \log \frac{M_1}{m_1} \leq \log \frac{M_0(4l + 1)}{m} = B_0 + \log(4l + 1).$$

□

Let us now estimate from above other geometric characteristics used in Theorem 4.4, namely, the length and total curvature of  $\Gamma$ . We have:

$$|\Gamma| = 4\pi l\nu < 4ln^{-2}; \quad |\kappa(\Gamma)| \leq 4\pi l.$$

We can now apply Theorem 4.4:

$$V_{\Gamma_1, l}(I) \leq c_{n, l} e^{\frac{5}{c''}(36n^2 + 4ln^{-2})n^2 \cdot 24}$$

where

$$c_{n, l} = (B_0 + \log(1 + 4l)) \left( \frac{4ln^{-2}}{\varepsilon} + 4\pi l + 1 \right).$$

By the Main Lemma,

$$B_0 < (1 - \log c'(H))A^2,$$

where  $A = e^{\frac{n^4}{c''}}$ . Elementary estimates imply:

$$c_{n, l} \leq ((1 - \log c'(H))A^3 e^l).$$

This implies (4.23).

Together, inequalities (4.21) - (4.23) imply Theorem B2.

## 4.7 Proof of Theorem A2 in Case 2 (near an infinite endpoint)

Here we prove Theorem A2 for a segment with one endpoint (say,  $b$ ) infinity (statement 2 mentioned at the beginning of the section).

**4.17 Proposition** *The integral  $I$  has an algebraic branching point at infinity of order  $n + 1$ .*

**Proof of Proposition 4.17.** Let  $S_R$  be the circle  $|t| = R$ ,  $R \geq 3$ ,  $\Gamma_R$  be the  $(n + 1)$  sheet cover of  $S_R$  with the base point  $-R$ . Consider the real ovals  $\gamma_t$  extended for  $t \in W$ . For any arc  $\Gamma' \subset \Gamma_R$  going from  $-R$  to  $t_\varphi = -Re^{i\varphi}$  let  $[\Delta_{\Gamma'}]$  be the class of all the covering homotopy maps  $\{H = -R\} \rightarrow \{H = t_\varphi\}$ . Let  $h$  be the highest homogeneous part of  $H$ . If  $H = h$ , then for any  $R$  the class  $[\Delta_{\Gamma'}]$  contains the simple rotation:

$$R_0 : (x, y) \mapsto (e^{\frac{i\varphi}{n+1}}x, e^{\frac{i\varphi}{n+1}}y)$$

In the general case, for  $R$  large enough the class  $[\Delta_{\Gamma'}]$  contains a map  $\Delta_{\Gamma'}$  close to the rotation. Let us prove this statement. To do this, consider the extension of the foliation  $H = \text{const}$  by complex level curves of  $H$  to the projective plane  $\mathbb{P}^2$  obtained by pasting the infinity line to the coordinate plane  $\mathbb{C}^2$ . The foliations  $H = \text{const}$  and  $h = \text{const}$  are topologically equivalent near infinity. More precisely, for any  $r > 0$  large enough there exists a homeomorphism  $\Phi$  of the complement  $\mathbb{P}^2 \setminus D_r$  ( $D_r$  is the ball of radius  $r$  centered at 0) onto a domain in  $\mathbb{P}^2$  that preserves the infinity line such that  $h \circ \Phi = H$ . This follows from the statements that the singularities of these foliations at infinity are the same and of the same topological type (nodes), and the holonomy mappings corresponding to circuits around these singularities in the infinity line are rotations  $t \mapsto e^{\frac{2\pi i}{n+1}}t$  in the transversal coordinate  $t = H^{\frac{1}{n+1}}$ . The last statement follows from the fact that for a generic  $C \in \mathbb{C}$

$$H(x, y)|_{x=Cy} = (\tilde{C}x)^{n+1}(1 + o(1)), \text{ as } x \rightarrow \infty, \tilde{C} \neq 0.$$

The homeomorphism  $\Phi$  is close to identity near infinity. For any  $r > 0$  there exists a  $T(r) > 0$  such that for any  $t$ ,  $|t| > T(r)$ ,  $S_t \cap D_r = \emptyset$ . The map  $\Delta_{\Gamma'}$  we are looking for is obtained from the map  $R_0$  corresponding to  $h$  by conjugation by the homeomorphism  $\Phi$ . By construction, its  $n + 1$ - iterate is identity.  $\square$

**Proof of Theorem A2 near infinity.** Let  $V$  be the Riemann surface of the integral  $I$ . Let  $\Gamma \subset V$  be the degree  $n + 1$  cover of the circle  $|t| = 3$  with the base point  $t_1 = +3$ . This is a closed curve on  $V$ . This curve is a boundary of a domain on  $V$  that covers a neighborhood of infinity. Let us denote this domain by  $V_\infty$ . We will estimate from above

$$N_\infty = \{t \in V_\infty \mid I(t) = 0\}.$$

This will give an upper estimate to the number of zeros of  $I$  on  $\sigma^+ = (3, +\infty)$  because  $\sigma^+ \subset V_\infty$ . We will use the argument principle in the form

$$N_\infty \leq \frac{1}{2\pi} V_\Gamma(I).$$

The variation in the right hand side will be estimated by Theorem 4.4. To apply this theorem we need to define all the entries like in the previous subsection.

Let  $\Gamma = \partial V_\infty$ . Without loss of generality we consider that  $I|_\Gamma \neq 0$  (one can achieve this by slight contraction of the circle  $|t| = 3$ ). Let  $K$  be the same as in the Main Lemma. Denote by  $U_0$  the set  $U$  from that lemma: both  $K$  and  $U_0$  are taken projected to the Riemann surface of the integral  $I$ . Let  $\varepsilon$  be the same as in (4.9). One has  $K \supset \Sigma$ , see 1.7, hence,  $K \supset \Gamma$ .

Let

$$U'' = \overline{K^\varepsilon}, \quad U' = \overline{K^{2\varepsilon}}, \quad U = \overline{K^{3\varepsilon}}.$$

By (1.11), the diameter condition (4.7) holds with

$$D = 36n^2 + 1.$$

The gap condition (4.6) for  $\Gamma, U'', U', U$  holds as well. The Bernstein index  $B = B_{U'', U}(I)$  may be easily estimated with the use of the same results that were used in the estimate of  $B_0 = B_{K, U_0}(I)$ . Indeed,

$$B = \log \frac{M'}{m'}, \quad M' = \max_{\overline{U}} |I|, \quad m' = \max_{\overline{U''}} |I|;$$

$$B_0 = \log \frac{M_0}{m}, \quad M_0 = \max_{\overline{U_0}} |I|, \quad m = \max_K |I|.$$

But  $K \subset U''$ ; hence,  $m' \geq m$ . On the other hand,

$$U = U_0 \cup \Gamma^{3\varepsilon}.$$

Each points of  $U$  may be connected to  $t_0$  by a path that satisfies the assumptions of Theorem 2.10. (Let  $M_0$  be the constant from the same theorem.) Hence, by Corollary 3.6,

$$M' = \max_U |I(t)| < n^4 M_0 = M'_1$$

In the proof of the Main Lemma we used the following inequalities:

$$M' \leq M'_1, \quad \log \frac{M'_1}{m} < (1 - \log c')A^2.$$

Hence,

$$B = \log \frac{M'}{m'} < \log \frac{M'_1}{m} < (1 - \log c')A^2.$$

This inequality will be substituted in (4.5). Another quantities from (4.5):

$$e^{\frac{5D}{\varepsilon}} \leq A^{4700},$$

$$|\Gamma| \leq 6\pi(n+1),$$

$$|\kappa(\Gamma)| \leq 2\pi(n+1).$$

Altogether, by Theorem 4.4, this implies Theorem A2, Case 2. □

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